Our goal in this lecture is to study the relative connectivity properties of the quotient spaces $PL(m)/O(m)$. Our basic observation is the following:

**Remark 1.** Let $K \subseteq \mathbb{R}^m$ be a closed subpolyhedron. Then the mapping space $(PL(m)/O(m))^K$ can be identified with the simplicial set $\text{Smooth}(K)$ of germs of smooth structures on $\mathbb{R}^m$ near $K$. This follows from the main result of the last lecture, together with the observation that the standard PL structure on $\mathbb{R}^m$ determines a constant map $\chi: \mathbb{R}^m \rightarrow BPL(m)$.

**Proposition 2.** Fix an integer $m \geq 0$. The following conditions are equivalent:

1. All homotopy fibers of the map $PL(m)/O(m) \rightarrow PL(m+1)/O(m+1)$ are $(m-1)$-connected.
2. All homotopy fibers of the map $BO(m) \rightarrow BO(m+1) \times BPL(m+1) BPL(m)$ are $(m-1)$-connected.
3. The following weak product structure theorem holds:
   
   $(\ast)$ Let $M$ be a PL manifold of dimension $m$, let $K \subseteq M$ be a closed subpolyhedron, and suppose we are given a smooth structure on $M \times \mathbb{R}$ which is the product of a smooth structure on $M$ with the standard smooth structure on $\mathbb{R}$ in a neighborhood of $K \times \mathbb{R}$. Then, after modifying the smooth structure by a suitable PD isotopy which is trivial in a neighborhood of $K \times \mathbb{R}$, we can arrange that the smooth structure on $M \times \mathbb{R}$ is the product of a smooth structure on $M$ with the standard smooth structure on $\mathbb{R}$.

**Proof.** We have a natural transformation of homotopy fiber sequences

$$
\begin{array}{cccc}
PL(m)/O(m) & \xrightarrow{\phi} & BO(m) & \xrightarrow{\psi} & BPL(m) \\
\downarrow & & \downarrow & & \\
PL(m+1)/O(m+1) & \xrightarrow{\theta} BO(m+1) \times BPL(m+1) BPL(m) & \xrightarrow{} & BPL(m).
\end{array}
$$

It follows that every homotopy fiber of $\phi$ is also a homotopy fiber of $\psi$, so the implication (2) $\Rightarrow$ (1) is clear. To prove the converse, it suffices to show that every homotopy fiber of $\psi$ is equivalent to a homotopy fiber of $\phi$. This will follow if the map $\theta$ is surjective on $\pi_0$. This surjectivity follows from the fiber sequence, since $BPL(m)$ is connected.

We now prove that (2) $\Rightarrow$ (3). In the situation of (3), the smooth structure on $M \times \mathbb{R}$ is classified by a map $M \times \mathbb{R} \rightarrow BO(m+1) \times PL(m+1)$ $PL(m)$. Finding a PD isotopy to a smooth structure on $M \times \mathbb{R}$ which is a product with $\mathbb{R}$ is equivalent to solving the lifting problem

$$
\begin{array}{cccc}
BO(m) & \xrightarrow{} & BPL(m) \\
\downarrow & & \\
M \times \mathbb{R} & \xrightarrow{} & BO(m) \times BPL(m) BPL(m).
\end{array}
$$
If we wish to do achieve this via an isotopy fixed near $K$, then we must solve instead a relative lifting problem of the form

$$
\begin{align*}
K \times \mathbb{R} & \longrightarrow BO(m) \\
M \times \mathbb{R} & \longrightarrow BO(m+1) \times_{BPL(m+1)} BPL(m).
\end{align*}
$$

This is a purely homotopy theoretic problem; we may therefore replace the inclusion $K \times \mathbb{R} \subseteq M \times \mathbb{R}$ by $K \subseteq M$. Since $M$ is a PL $m$-manifold, it can be obtained from $K$ by successive cell attachments where the cells have dimension $\leq m$. Working cell-by-cell, we are reduced to solving lifting problems of the form

$$
\begin{align*}
\partial D^k & \longrightarrow BO(m) \\
D^k & \longrightarrow BO(m+1) \times_{BPL(m+1)} BPL(m)
\end{align*}
$$

where $D^k$ indicates a disk of dimension $\leq k$. The obstruction to solving such a problem is equivalent to the vanishing of a class in $\pi_{k-1}$ of a homotopy fiber $F$ of $\psi$. This class automatically vanishes by virtue of our assumption that $F$ is $(m-1)$-connected.

We now prove that $(3) \Rightarrow (1)$. We must show that every lifting problem of the form

$$
\begin{align*}
\partial D^k & \longrightarrow PL(m)/O(m) \\
D^k & \longrightarrow PL(m+1)/O(m+1)
\end{align*}
$$

has a solution, provided that $k \leq m$. In this case, we can choose a PL embedding of $\partial D^k$ into $\mathbb{R}^m$ and obtain an equivalent lifting problem

$$
\begin{align*}
\partial D^k \times \mathbb{R} & \longrightarrow PL(m)/O(m) \\
\mathbb{R}^{m+1} & \longrightarrow PL(m+1)/O(m+1).
\end{align*}
$$

The diagram determines a smoothing of $\mathbb{R}^{m+1}$ which is a product smoothing in a neighborhood of $\partial D^k \times \mathbb{R}$, and a solution to the indicated lifting problem is equivalent to giving a PD isotopy (fixed near $\partial D^k \times \mathbb{R}$) to a product smoothing.

**Remark 3.** If the equivalent conditions of Proposition 2 are satisfied, then the map $PL(m)/O(m) \rightarrow PL(m+1)/O(m+1)$ is surjective on $\pi_0$ for $m \geq 0$. Since $PL(0)/O(0) = \ast$ is connected, we it follows by induction that $PL(m)/O(m)$ is connected for each $m$. In other words, Euclidean space $\mathbb{R}^m$ admits a unique smooth structure compatible with its standard PL structure, up to PD isotopy.

The connectivity estimate given in Proposition 2 is not the best possible. We now describe how to do a little better. We need a variation on the main result of the last lecture, which applies to manifolds with boundary.

**Variant 4.** Let $M$ be a PL $(m+1)$-manifold with boundary $\partial M$. We can define the notion of a smoothing of $M$ as before. Smoothings of $M$ can be organized into a simplicial set $\text{Smooth}(M)$. Every smoothing of $M$ determines a smoothing of the boundary of $M$; this is given by a Kan fibration $\text{Smooth}(M) \rightarrow \text{Smooth}(\partial M)$. Given a smooth structure on the boundary of $M$, we denote the fiber of this map by $\text{Smooth}(M; \partial)$. Given
such a smoothing of \( \partial M \), we get a map \( \partial M \to BO(m) \). Then, up to homotopy, smoothings of \( M \) compatible with this smooth structure on \( \partial M \) are given by solutions to the lifting problem

\[
\begin{array}{c}
\partial M \\
\downarrow \\
M \\
\downarrow \\
BO(m+1) \\
\downarrow \\
BPL(m+1).
\end{array}
\]

Smoothings of \( M \) itself (without boundary data) can be identified with solutions to the lifting problem of pairs

\[
\begin{array}{c}
(BO(m+1), BO(m)) \\
\downarrow \\
(M, \partial M) \\
\downarrow \\
(BPL(m+1), BPL(m)).
\end{array}
\]

**Notation 5.** Fix an integer \( m \geq 0 \). We let \( \Delta^{PL}_m \) denote the homotopy fiber product

\[
BPL(m) \times^{h}_{BPL(m+1)} BPL(m) = BPL(m) \times_{BPL(m+1)(\partial)} BPL(m+1)^{[0,1]} \times_{BPL(m+1)^{[1]}} BPL(m).
\]

Similarly, define \( \Delta^{O}_m \) to be the homotopy fiber product

\[
BO(m) \times^{h}_{BO(m+1)} BO(m) = BO(m) \times_{BO(m+1)(\partial)} BO(m+1)^{[0,1]} \times_{BO(m+1)^{[1]}} BO(m).
\]

We have a Kan fibration \( \Delta^{O}_m \to \Delta^{PL}_m \). For every PL \( m \)-manifold \( M \), the tangent microbundle to \( M \times [0,1] \) and its boundary determines a map \( M \to \Delta^{PL}_m \). According to Variation 4, we can identify smoothings of \( M \times [0,1] \) with solutions to the lifting problem

\[
\begin{array}{c}
\Delta^{O}_m \\
\downarrow \\
M \\
\downarrow \\
\Delta^{PL}_m.
\end{array}
\]

The proof of Proposition 2 adapts without essential change to show the following:

**Proposition 6.** Fix an integer \( m \geq 0 \). The following conditions are equivalent:

1. Let \( F \) denote the homotopy fiber of the map \( \Delta^{O}_m \to \Delta^{PL}_m \). Then all \( PL(m)/O(m) \to F \) are \((m-1)\)-connected.

2. All homotopy fibers of the map \( BO(m) \to \Delta^{O}_m \times_{\Delta^{PL}_m} BPL(m) \) are \((m-1)\)-connected.

3. The following strong product structure theorem holds:

   (*) Let \( M \) be a PL manifold of dimension \( m \), let \( K \subseteq M \) be a closed subpolyhedron, and suppose we are given a smooth structure on \( M \times [0,1] \) which is the product of a smooth structure on \( M \) with the standard smooth structure on \([0,1]\) in a neighborhood of \( K \times \mathbb{R} \). Then, after modifying the smooth structure by a suitable PD isotopy which is trivial in a neighborhood of \( K \times [0,1] \), we can arrange that the smooth structure on \( M \times [0,1] \) is the product of a smooth structure on \( M \) with the standard smooth structure on \([0,1]\).

**Remark 7.** Let \( F \) be as in Proposition 6. Then the homotopy fibers of the map \( PL(m)/O(m) \to F \) can be identified with path spaces in the space in homotopy fibers of the map \( \psi : PL(m)/O(m) \to PL(m+1)/O(m+1) \). Consequently, if we grant that the homotopy fibers of \( \psi \) are nonempty (which follows from
Proposition 2 if \( m \geq 0 \), then Proposition 6 asserts that the homotopy fibers of \( \psi \) are \( m \)-connected. This is a slightly better connectivity estimate than we get from Proposition 2 itself, which is why the geometric assertion of part (3) of Proposition 6 is called the \textit{strong} product structure theorem to contrast it with the corresponding \textit{weak} product structure theorem of Proposition 2. However, the terminology is slightly misleading: Proposition 6 does not quite formally imply Proposition 2, since it does not guarantee that the homotopy fibers of \( \psi \) are nonempty. This missing strength is equivalent to the assertion of Remark 3: we need to know that every smooth structure on \( \mathbb{R}^m \) is PD isotopic to the product with \( \mathbb{R} \) of a smooth structure on \( \mathbb{R}^{m-1} \), and thus (using induction on \( m \)) PD isotopic to the standard smooth structure on \( \mathbb{R}^m \).