Recall that our goal is to prove the following result:

**Theorem 1.** Let $M$ be a PL manifold. The above construction determines a homotopy equivalence from the simplicial set $\text{Smooth}(M)$ of smooth structures on $M$ to the simplicial set

$$BO(m)^M \times_{BPL(m)^M} \{\chi\}$$

of liftings of $\chi$. In particular, $M$ admits a smoothing if and only if there exists a commutative diagram

$$\begin{array}{ccc}
BO(m) & \xrightarrow{L} & BPL(m) \\
\downarrow & & \downarrow \\
M & \xrightarrow{\chi} & BO(m) \times BPL(m).
\end{array}$$

To prove Theorem 1, it will be convenient to formulate a more local version. For every open subset $U \subseteq M$, let $\text{Smooth}(U)$ denote the simplicial set of smooth structures on $U$. The assignment $U \mapsto \text{Smooth}(U)$ defines a sheaf of simplicial sets on $M$. We can extend the definition of this sheaf to closed subpolyhedra $K \subseteq M$ by the formula $\text{Smooth}(K) = \lim_{\rightarrow} K \subseteq U \text{Smooth}(U)$. We now have the following generalization of Theorem 1:

**Theorem 2.** Let $M$ be a PL manifold and $K \subseteq M$ a closed subpolyhedron. Then the above construction determines a homotopy equivalence from the simplicial set $\text{Smooth}(K)$ of smooth structures on $M$ to the simplicial set

$$BO(m)^K \times_{BPL(m)^K} \{\chi|K\}$$

of liftings of $\chi|K$.

We observe that Theorem 2 is trivial in the case where $K$ is a point: in this case, the map $\text{Smooth}(K) \to BO(m) \times_{BPL(m)} *$ is an isomorphism of simplicial sets.

In the statement of Theorem 2, the right hand side has a description in terms of sections of fibrations, and is thus under good homotopy-theoretic control. To prove Theorem 2, we will need a similar understanding of the left hand side. This is furnished by the following fact, which is the main objective of this lecture:

**Proposition 3 (Flexibility).** Let $K \subseteq K'$ be compact subpolyhedra of $M$. Then the restriction map $\text{Smooth}(K') \to \text{Smooth}(K)$ is a Kan fibration.

Note that $\text{Smooth}(K') = \lim_{\rightarrow} K' \subseteq V \text{Smooth}(V)$. Since a direct limit of Kan fibrations is a Kan fibration, it will suffice to prove that each of the maps $\text{Smooth}(V) \to \text{Smooth}(K)$ is a Kan fibration. Replacing $M$ by $V$, we are reduced to proving the following:

**Proposition 4.** Let $K$ be a compact subpolyhedron of $M$. Then the restriction map $\text{Smooth}(M) \to \text{Smooth}(K)$ is a Kan fibration.
We must show that every lifting problem of the form

\[\Lambda_i^n \rightarrow \text{Smooth}(M)\]
\[\Delta^n \rightarrow \text{Smooth}(K)\]

has a solution. The top map determines a PD homeomorphism \(\Lambda_i^n \times M \to N\), where \(N\) is a smooth fiber bundle over \(\Lambda_i^n\). Since the horn \(\Lambda_i^n\) is contractible, we can write \(N = \Lambda_i^n \times N_0\), where \(N_0\) is a smooth manifold. The bottom map determines an open subset \(U\) of \(M \times \Delta^n\) containing \(K \times \Delta^n\) and a PD homeomorphism \(U \to W\), where \(W\) is a smooth fiber bundle over \(\Delta^n\) whose restriction to \(\Lambda_i^n\) can be identified with an open subset of \(\Lambda_i^n \times N_0\). Since \(\Delta^n\) is trivial, we can write \(W = W_0 \times \Delta^n\), where \(W_0\) is a smooth manifold. Unwinding everything, we have the following data:

1. A PD family \(\{f_v : M \to N_0\}_{v \in \Lambda_i^n}\) of PD homeomorphisms.
2. A PD homeomorphism \(g : U \simeq \Delta^n \times W_0\), compatible with the projection to \(\Delta^n\).
3. A smooth family of open embeddings \(\{h_v : W_0 \to N_0\}_{v \in \Lambda_i^n}\) such that the following diagrams commute:

\[\begin{array}{ccc}
U \times \Delta^n & \{v\} & M \\
| & \downarrow{g_v} & \downarrow{f_v} \\
W_0 & \downarrow{h_v} & N_0.
\end{array}\]

Let \(B \subseteq N_0\) be a compact set containing the image of \(K \times \Delta^n\) in its interior. Enlarging \(B\), we may suppose that \(B\) is a smooth submanifold with boundary of \(N\) with codimension zero. Fix a point \(0 \in \Lambda_i^n\). Using the parametrized isotopy extension theorem (in the smooth category), we can find a smooth family of diffeomorphisms \(\{h_v' : M \to M\}_{v \in \Lambda_i^n}\) such that \((h_v'h_0)B = h_vB\). Replacing \(h_v\) by \(h_v'\) and \(f_v\) by \(h_v'^{-1}f_v\), we can assume that \(h_v\) is constant on the interior \(B\). Replacing \(W_0\) by the interior of \(B\) and shrinking \(U\), we may assume that \(h_v\) is actually constant. We may therefore identify \(W_0\) with an open subset of \(N_0\).

To prove the existence of the desired extension, it will suffice to show that we can extend \(f_v\) to a PD family of PD homeomorphisms \(\{f_v' : M \to N_0\}_{v \in \Delta^n}\), such that the families \(\{f_v'\}\) and \(g\) agree in a neighborhood of \(K\). Enlarging \(K\), it will suffice to guarantee that we can arrange these maps to agree on \(K\) itself. Choose a PL homeomorphism \(\Delta^n \simeq C \times [0, 1]\), where \(C = \Lambda_i^n\), and view \(\{g_v\}_{v \in \Delta^n}\) as a two-parameter family \(\{g_{c,t}\}_{c \in C, t \in [0, 1]}\).

Note that \(f_v\) and \(g\) determine a polyhedral structure \(S\) on

\[(N_0 \times C) \coprod_{g(P) \times [0, 1] \{0\}} g(P)\]

where \(P\) is any closed subpolyhedron of \(U\). Choose \(P\) to contain \(K \times \Delta^n\). Our existence results for triangulations show that we can find a Whitehead compatible triangulation of \(N_0 \times C \times [0, 1]\) which is compatible with the projection to \(C \times [0, 1]\) and agrees with \(S\) near \(N_0 \times C \times \{0\}\) and near \(g(K \times C \times [0, 1])\). Since the projection \(\pi : N_0 \times C \times [0, 1] \to C \times [0, 1]\) is a fiber bundle in the smooth category, it is also a fiber bundle in the PL category, and can therefore be identified with \(\pi^{-1}(C \times \{0\}) \times [0, 1] \cong M \times C \times [0, 1]\). Using the parametrized isotopy extension theorem (in the PL category), we can adjust this identification so that it agrees with \(g\) on \(K \times C \times [0, 1]\). This provides the desired extension \(\{f_{c,t}\}_{c \in C, t \in [0, 1]}\) of \(\{f_c\}_{c \in C}\) and completes the proof.