Our goal in this lecture is to carry out the main step in the proof of the Kister-Mazur theorem describing the relationship between microbundles and $\mathbb{R}^n$-bundles. Namely, we will prove the following:

**Theorem 1.** Let $\text{Emb}(\mathbb{R}^n)$ denote the simplicial set of open embeddings from $\mathbb{R}^n$ to itself (so a $k$-simplex of $\text{Emb}(\mathbb{R}^n)$ is an open embedding $j: \mathbb{R}^n \times \Delta^k \to \mathbb{R}^n \times \Delta^k$ which commutes with the projection to $\Delta^k$), and let $\text{Homeo}(\mathbb{R}^n) \subseteq \text{Emb}(\mathbb{R}^n)$ denote the simplicial subset of homeomorphisms from $\mathbb{R}^n$ to itself (so that a $k$-simplex of $\text{Homeo}(\mathbb{R}^n)$ is a $k$-simplex of $\text{Emb}(\mathbb{R}^n)$ for which the map $j$ is a homeomorphism). Then the inclusion $i: \text{Homeo}(\mathbb{R}^n) \subseteq \text{Emb}(\mathbb{R}^n)$ is a homotopy equivalence of Kan complexes.

**Remark 2.** We can also define topological spaces parametrizing homeomorphisms or open embeddings from $\mathbb{R}^n$ to itself: Theorem 1 is equivalent to the assertion that the inclusion between these topological spaces is a weak homotopy equivalence.

**Remark 3.** We can also define simplicial sets which parametrize PL embeddings and PL homeomorphisms from $\mathbb{R}^n$ to itself. Theorem 1 continues to hold in this case, using essentially the same proof that we will give below.

The main step in the proof of Theorem 1 is to establish that $i$ is a surjection on $\pi_0$. In other words, every open embedding $f: \mathbb{R}^n \to \mathbb{R}^n$ is isotopic to a homeomorphism of $\mathbb{R}^n$ with itself. In fact, we will prove something more precise:

**Proposition 4.** Let $f$ be an open embedding from $\mathbb{R}^n$ to itself. Then there exists an isotopy $F_i$ from $f = F_0$ to a homeomorphism $f = F_1$. Moreover, this isotopy can be chosen to be constant on the unit ball $B(1)$ of $\mathbb{R}^n$.

**Notation 5.** For every positive real number $r$, let $B(r) = \{x \in \mathbb{R}^n : |x| < r\}$ be the open ball of radius $r$ around the origin (in giving the PL version of this proof, it is convenient to replace $B(r)$ by an open cube).

Here is the rough idea of the proof. The obstruction to an open embedding being a homeomorphism is that it might not be surjective. Our objective, therefore, is to use an isotopy to modify $f$ so that its image becomes larger and larger. More precisely, we will construct a sequence of open embeddings

$$f^1, f^2, \ldots : \mathbb{R}^n \to \mathbb{R}^n$$

and a sequence of isotopies $h^i : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ so that the following conditions are satisfied:

1. The map $f^1 = f$.
2. For each $i$, the map $h^i$ is an isotopy from $f^i = h^i_0$ to $f^{i+1} = h^i_1$, which is constant on the open ball $B(i)$.
3. For $i > 1$, we have $B(i) \subseteq f^i B(i)$. 

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Assuming that we can meet these requirements, we can define a homeomorphism \( f' : \mathbb{R}^n \to \mathbb{R}^n \) by the formula \( f'(x) = f(x) \) for any \( i \geq |x| \). We get an isotopy from \( f \) to \( f' \) by concatenating the isotopies \( h^1, h^2 \), and so forth (this concatenation is well-defined since almost all of the isotopies \( h^i \) are constant on any given compact subset of \( \mathbb{R}^n \)).

To begin, we may assume without loss of generality that \( f(0) = 0 \) (otherwise, we can reduce to this case by conjugating by a relevant translation). Since \( f \) is an open embedding, the image \( f(B(1)) \) contains an open ball \( B(\varepsilon) \) for some real number \( \varepsilon > 0 \). Since \( f \) is continuous, there exists a positive real number \( \delta < 1 \) such that \( f(B(\delta)) \subseteq B(\frac{\varepsilon}{2}) \).

To construct our isotopies \( h^i \), we will need the following basic building blocks:

**Notation 6.** For every pair of real numbers \( r < s \), we fix an isotopy \( H(r, s)_t : \mathbb{R}^n \to \mathbb{R}^n \) from \( \text{id}_{\mathbb{R}^n} \) to \( H(r, s)_1 \) with the following properties:

(i) The isotopy \( H(r, s)_t \) is trivial on \( B(\frac{\varepsilon}{2}) \) and supported in a compact subset of \( B(s + 1) \).

(ii) The map \( H(r, s)_1 \) restricts to a homeomorphism \( B(r) \to B(s) \).

We now proceed with the construction of the sequence \( \{f^i\} \). Assume that \( f^i \) has already been constructed. We wish to construct an isotopy \( h^i \) from \( f^i \) to another map \( f^{i+1} \), which is constant on \( B(i) \). First, we define a homeomorphism \( c \) (for “contraction”) from \( \mathbb{R}^n \) to itself as follows:

\[
c(x) = \begin{cases} 
  x & \text{if } x \notin f^i(\mathbb{R}^n) \\
  f^i(H(\delta, i)^{-1}(y)) & \text{if } x = f^i(y).
\end{cases}
\]

Since \( f^i = f \) on \( B(1) \) and \( f \) carries \( B(\delta) \) into \( B(\frac{\varepsilon}{2}) \), we deduce that \( c(f^i(x)) \in B(\frac{\varepsilon}{2}) \) if \( x \in B(i) \). Note that \( c \) is the identity outside a compact set, which we can take to be contained in \( B(N_i) \) for some \( N_i \gg i + 1 \).

We now define \( h^i_1 \) by the formula

\[
h^i_1 = c^{-1} \circ H(\epsilon, N_i)_t \circ c \circ f^i.
\]

It is clear that \( h^i_1 \) is an isotopy from \( f^i = h^0_1 \) to another map \( f^{i+1} = h^1_1 \). Moreover, since \( H(\epsilon, N_i)_t \) is the identity on \( B(\frac{\varepsilon}{2}) \) and \( c \circ f^i \) carries \( B(i) \) into \( B(\frac{\varepsilon}{2}) \), we deduce that \( h^i_1 \) is constant on \( B(i) \). It remains only to verify that \( f^{i+1}B(i + 1) \) contains \( B(i + 1) \). In fact, we claim that \( f^{i+1}B(i + 1) \) contains \( B(N_i) \). Since \( c \) is supported in \( B(N_i) \), it suffices to show that \( (cf^{i+1})B(i + 1) = (H(\delta, i)^{-1}_1 \circ c \circ f^i)B(i) \) contains \( B(N_i) \). For this, it suffices to show that \( (c \circ f^i)B(i) \) contains \( B(i) \subseteq f^iB(i + 1) \). This is clear, since \( H(\delta, i)^{-1}_1 \) induces a homeomorphism of \( B(i + 1) \) with itself. This completes the proof of Proposition 4.

**Remark 7.** In the above construction, each of the isotopies \( h^i \) is obtained by composing \( f^i \) with a 1-parameter family \( c^{-1} \circ H(\epsilon, N_i)_t \circ c \circ f^i \) of homeomorphisms from \( \mathbb{R}^n \) to itself. It follows that if the original map \( f \) is already a homeomorphism, then the isotopy \( F_t \) that we construct will be a path through the space of homeomorphisms.

Suppose now that we are given not a single open embedding \( f : \mathbb{R}^n \to \mathbb{R}^n \), but a family of open embeddings \( f : \mathbb{R}^n \times \Delta \to \mathbb{R}^n \times \Delta \) (compatible with the projection to \( \Delta \)), where \( \Delta \) is some parameter space. We might try to apply the above construction to each of the induced maps \( \{f_{v,t} : \mathbb{R}^n \to \mathbb{R}^n\}_{v \in \Delta} \) to produce a family of isotopies \( \{F_{v,t} : \mathbb{R}^n \to \mathbb{R}^n\}_{(v,t) \in \Delta \times [0,1]} \). We must be careful, since our construction depended on several choices. First of all, we needed to choose \( \epsilon \) such that \( f_{v,1} \) contains the open ball \( B(\epsilon) \). We note that \( f(B(1) \times \Delta) \) is an open neighborhood of \( \{0\} \times \Delta \) in \( \mathbb{R}^n \times \Delta \), which will contain some product neighborhood \( B(\epsilon) \times \Delta \) provided that \( \Delta \) is compact. We also needed to choose a constant \( \delta \) such that \( f_{v,1}B(\delta) \subseteq B(\frac{\varepsilon}{2}) \). Again, if \( \Delta \) is compact, then a sufficiently small real number \( \delta \) will work for all \( f_{v} \)'s simultaneously. Finally, to construct each \( h^i_1 \), we needed to choose \( N_i \gg i + 1 \), so that the relevant contraction \( c_{v} \) has compact support in \( B(N_i) \). The support of \( c_{v} \) is contained in \( f_{v,1}B(i + 1) \). If \( \Delta \) is compact, the image \( f_{v}B(i + 1) \times \Delta \) will be contained in a compact subset of \( \mathbb{R}^n \times \Delta \), which is in turn contained in \( B(N_i) \times \Delta \) for sufficiently large \( N_i \). Consequently, we get the following more refined version of Proposition 4:
Proposition 8. Let $\Delta$ be a compact topological space (for example, a simplex), and suppose we are given an open embedding $f : \mathbb{R}^n \times \Delta \to \mathbb{R}^n \times \Delta$ which is compatible with the projection to $\Delta$. Then there exists an isotopy $F : \mathbb{R}^n \times \Delta \times [0, 1] \to \mathbb{R}^n \times \Delta \times [0, 1]$ with the following properties:

1. The map $F_0$ coincides with $f$.
2. The map $F_1$ is a homeomorphism.
3. The isotopy $F$ is constant along $B(1) \times \Delta$.
4. If $f_v$ is already a homeomorphism for some $v \in \Delta$, then the isotopy $F_v : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n \times [0, 1]$ consists of homeomorphisms.

We can now prove Theorem 1. The proof is based on the following criterion for detecting homotopy equivalences:

Proposition 9. Let $i : K \subseteq K'$ be an inclusion of Kan complexes. Then $i$ is a homotopy equivalence if and only if the following condition is satisfied:

$\ast$ For every $n$-simplex $\sigma$ of $K'$ whose boundary belongs to $K$, there exists a homotopy $h : \Delta^n \times \Delta^1 \to K'$ such that $h|\Delta^n \times \{0\} = \sigma$, $h|\Delta^n \times \{1\}$ factors through $K$, and $h|\partial \Delta^n \times \Delta^1$ factors through $K$.

Roughly speaking, the simplex $\sigma$ is a typical representative of a class in $\pi_{n-1}$ of the homotopy fiber of the inclusion $K \to K'$, and condition (\ast) guarantees that any such class is trivial.

Theorem 1 follows immediately from Proposition 9 and Proposition 8.

In the next lecture, we will discuss the consequences of Theorem 1 for the classification of microbundles.