Recall that our goal is to prove the following:

**Theorem 1.** Let $X$ be a Poincare pair of dimension $n \geq 5$, $\zeta$ a stable PL bundle on $X$, and $f : M \to X$ a degree one normal map, where $M$ is a PL manifold. Let $\sigma_f^q \in \Omega^\infty \mathbb{L}^q(X, \zeta_X)$ be the relative signature of $f$, and suppose we are given a path $p$ from $\sigma_f^q$ to the base point of $\Omega^\infty \mathbb{L}^q(X, \zeta_X)$. (We can identify such a path with a Lagrangian in the Poincare object representing $\sigma_f^q$, which is well-defined up to bordism). Then there exists a $\Delta^1$-family of degree one normal maps $F : B \to X \times \Delta^1$, where $B$ is a bordism from $M = F^{-1}(X \times \{0\})$ to a PL manifold $N = F^{-1}(X \times \{1\})$ such that $F$ induces a homotopy equivalence $f' : N \to X$. Moreover, we can arrange that $F$ determines a path from $\sigma_f^q$ to $\sigma_{f'}^q = 0$ which is homotopic to $p$.

In the last lecture, we introduced the technique of surgery as a method of producing normal bordisms from $M$ to other PL manifolds (equipped with degree one normal maps to $X$). Moreover, we saw how to use the method of surgery to reduce Theorem 1 to the special case where $f : M \to X$ induces an equivalence of fundamental groupoids. We may further assume without loss of generality that $X$ (and therefore also $M$) are connected. Let us fix a base point of $X$, allowing us to define a fundamental group $G = \pi_1 X$. Let $\tilde{X}$ denote the universal cover of $X$ and let $\tilde{M} = M \times_X \tilde{X}$ be the corresponding universal cover of $M$, so that $G$ acts on $\tilde{X}$ and $\tilde{M}$ by deck transformations.

The spherical fibration $\zeta_X$ is classified by a map $X \to \text{Pic}(S)$, which induces a map

$$G = \pi_1 X \to \pi_1 \text{Pic}(S) = \pi_0 \text{GL}_1(S) = \text{GL}_1(\pi_0 S) = \text{GL}_1(\mathbb{Z}) = \{ \pm 1 \},$$

which we will denote by $\epsilon$. This homomorphism vanishes if and only if $\zeta_X$ is orientable with respect to ordinary homology (that is, if and only if $\epsilon$ and $\zeta$ is a constant sheaf). Let $Z[\pi_1 X] = Z[G]$ be the group algebra of $G$. Then $Z[G]$ admits an involution, given by $g \mapsto \epsilon(g) g^{-1}$. Let $Q^*, Q^\natural : (\text{LMod}_{Z[G]}^{fp})^{op} \to \text{Sp}$ be the quadratic functors given by

$$Q^q(M) = \text{Mor}_{Z[G]}(-Z[G] (M \wedge M, Z[G]))_{h\Sigma_2}$$

$$Q^\natural(M) = \text{Mor}_{Z[G]}(-Z[G] (M \wedge M, Z[G])^{h\Sigma_2}.$$

Using the $\pi$-$\pi$ theorem, we can identify $\Omega^\infty \mathbb{L}^q(X, \zeta_X)$ with $\Omega^\infty + n \mathbb{L}^q(Z[G]) \simeq L(\text{LMod}_{Z[G]}^{fp}, \Omega^n Q^\natural)$.

Let us attempt to describe the invariant $\sigma_f^q$ more explicitly in these terms. The visible symmetric signatures $\sigma^*_X$ and $\sigma^*_M$ determine Poincare objects of $(\text{LMod}_{Z[G]}^{fp}, \Omega^n Q^\natural)$. Unwinding the definitions, we see that these objects are given concretely by the duals of the $Z[G]$-modules given by $C_*(\tilde{X}; Z)$ and $C_*(\tilde{M}; Z)$ (note that each of these is a finitely presented $Z[G]$-module, since $\tilde{X}$ and $\tilde{M}$ admit cell decompositions which are invariant under $G$, whose cells break up into finitely many free $G$-orbits). Using Poincare duality, we see that both of these objects are self-dual up to a shift; more precisely, the relevant Poincare objects are represented by $\Sigma^{-n} C_*(\tilde{X}; Z)$ and $\Sigma^{-n} C_*(\tilde{M}; Z)$.

**Remark 2.** Using Poincare duality on the noncompact manifold $\tilde{M}$, we can identify $\Sigma^{-n} C_*(\tilde{M}; Z)$ with $C_c^*(\tilde{M}; Z)$, where the subscript indicates that we take compactly supported cochains. This identification
is not quite $G$-equivariant, since the action of $G$ on $\tilde{M}$ may not preserve orientations (the failure of the $G$-action to preserve orientations is codified by the homomorphism $\epsilon : G \to \{\pm 1\}$). Informally speaking, the symmetric bilinear form on $C_*^G(\tilde{M}; \mathbb{Z})$ is easy to describe: it carries a pair of compactly supported $\mathbb{Z}$-valued cochains $u$ and $v$ to the sum

$$\sum_{g \in G} (u \cup g(v))[M] \in \mathbb{Z}[G].$$

Here the condition that $u$ and $v$ both have compact support guarantees that the sum on the left hand side is indeed finite.

We have seen that the degree one map $f : M \to X$ determines a homotopy equivalence

$$\Sigma^{-n}C_*(\tilde{M}; \mathbb{Z}) \simeq V \oplus \Sigma^{-n}C_*(\tilde{X}; \mathbb{Z})$$

for some (finitely presented) $\mathbb{Z}[G]$-module spectrum $V$. Moreover, there is a point $q \in \Sigma^{-n}Q^p(V)$ such that $(V, q)$ is a Poincare object of $\text{LMod}^p_{\mathbb{Z}[G]}$, representing the relative signature $\sigma_{Jq}^p$.

Now suppose we are given a normal surgery datum in $M$. We have seen that the degree one map $f : M \to X$ determines a homotopy equivalence

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Now suppose we are given a normal surgery datum in $M$, giving in particular a codimension zero embedding $\alpha : S^p \times D^{p+1} \hookrightarrow M$. This determines a normal bordism from $M$ to another PL manifold $N$ equipped with a degree one normal map $f : N \to X$, hence a bordism between the Poincare objects representing $\sigma_{Jq}^p$ and $\sigma_{Jq}^p$. The latter bordism is given by an algebraic surgery along some map of $\mathbb{Z}[G]$-module spectra $u : \Sigma^{-n}K \to V$. Let $B(\alpha)$ denote the trace of the surgery along $\alpha$ and let $\tilde{B}(\alpha) = B(\alpha) \times_X \tilde{X}$. Then $\text{cofib}(u)$ is the $\mathbb{Z}[G]$-module underlying relative signature associated to $B(\alpha)$ (as a normal bordism); that is, we have

$$\Sigma^{-n}C_*(\tilde{X}; \mathbb{Z}) \oplus \text{cofib}(u) \simeq \Sigma^{-n}C_*(\tilde{B}(\alpha); \mathbb{Z}).$$

It follows that $\Sigma^{-n}K \simeq \text{fib}(V \to \text{cofib}(u)) \simeq \text{fib}(\Sigma^{-n}C_*(\tilde{M}; \mathbb{Z}) \to \Sigma^{-n}C_*(\tilde{B}(\alpha); \mathbb{Z})$. We have a homotopy pushout diagram of spaces

$$\begin{array}{ccc} S^p & \longrightarrow & D^{p+1} \\ \downarrow & & \downarrow \\ M & \longrightarrow & B(\alpha) \end{array}$$

which lifts to a homotopy pushout diagram of $G$-spaces

$$\begin{array}{ccc} S^p \times G & \longrightarrow & D^{p+1} \times G \\ \downarrow & & \downarrow \\ \tilde{M} & \longrightarrow & \tilde{B}(\alpha). \end{array}$$

It follows that $K$ is equivalent to the homotopy fiber of the map of the map $C_*(S^p \times G; \mathbb{Z}) \to C_*(D^{p+1} \times G; \mathbb{Z})$, which is homotopy equivalent to $\Sigma^p \mathbb{Z}[G]$. The map $u : \Sigma^{-n}K \to V$ is classified up to homotopy by an element of $\pi_{p-n}V$, which we can regard as a direct summand of $\pi_p C_*(\tilde{M}; \mathbb{Z}) \simeq H_p(\tilde{M}; \mathbb{Z})$. The above calculation shows that this homology class if the Hurewicz image of the class in $\pi_p \tilde{M}$ determined by a choice of lift of the map $\alpha_0 : S^p \to M$ determined by the surgery datum $\alpha_0$.

Remark 3. In the above discussion, the module $K \simeq \Sigma^p \mathbb{Z}[G]$ is determined by the choice of dimension $p$, and the map $u : \Sigma^{-n}K \to V$ is determined by the homotopy class of the map $\alpha_0 : S^p \to M$ (and a nullhomotopy $h$ of the composite map $S^p \to M \to X$). To perform algebraic surgery on the Poincare object $(V, q)$, we need more: namely, a nullhomotopy of the restriction $q|\Sigma^{-n}K$. This choice of nullhomotopy depends on additional geometric data: the fact that $\alpha_0$ is an embedding, and a choice of trivial normal bundle to $\alpha_0$ compatible with $h$. 

2
The key step in the proof of Theorem 1 is the following, which asserts that there is a sufficient supply of normal surgery data:

**Theorem 4.** Let \( f : M \to X \) be as in Theorem 1. Assume that \( M \) and \( X \) are connected and that \( f \) induces an isomorphism \( \pi_1 M \cong \pi_1 X \cong G \), and let \((V,q)\) be defined as above. Assume that \( f \) is \( p \)-connected, that we are given a map \( u : \Sigma^{p-n} Z[G] \to V \) a nullhomotopy of \( q \Sigma^{p-n} Z[G] \), so that (algebraic) surgery along \( u \) determines a bordism bordism from \((V,q)\) to another Poincare object \((V',q')\). Then this (algebraic) bordism can be obtained by performing (geometric) surgery with respect to a normal surgery datum \( \alpha : S^p \times D^{q+1} \to M \).

**Remark 5.** In the situation of Theorem 4, the relevant surgery does not change the fundamental group of \( M \). the relevant \( p \)-surgeries do not change the fundamental group of \( M \). Suppose we are given an embedding \( \alpha : S^p \times D^{q+1} \to M \) (where \( p + q + 1 = n \)). The manifold \( M^o \) obtained from \( M \) by removing the interior of the image of \( \alpha \) is homotopy equivalent to \( M \times S^p \), which differs from \( M \) in codimension \( q + 1 = n - p \). General position arguments show that this procedure does not change the fundamental group of \( M \) provided that \( n - p \ge 3 \). This condition is clearly satisfied when \( n \ge 5 \) and \( p \le \frac{n}{2} \). Surgery along \( \alpha \) produces a new manifold \( M' \), which is obtained as a pushout

\[
M^o \coprod_{S^p \times S^q} D^{q+1} \times S^q.
\]

Since \( q = n - p - 1 \ge 2 \), the sphere \( S^q \) is simply connected. It follows from van Kampen’s theorem \( \pi_1 M^o \to \pi_1 M' \) is surjective. Since the composite map \( \pi_1 M^o \to \pi_1 M' \to \pi_1 X \) is injective, we deduce that \( \pi_1 M' \cong \pi_1 M^o \).

Our goal for the remainder of this lecture is to explain how to deduce Theorem 1 from Theorem 4. To this end, let us suppose that we are given an arbitrary Lagrangian in \((V,q)\), given by a map \( L \to V \) and a nullhomotopy of \( qL \). We would like to show that the Lagrangian \( L \) can be obtained by a sequence of normal surgeries on the PL manifold \( M \). Before we can make this assertion, we may need to modify the choice of Lagrangian \( L \). Recall that the data of \( V \) together with the Lagrangian \( L \) can be identified with a quadratic object \((W,q')\) of \((L\text{Mod}_{Z[G]}^p, \Sigma^{-n-1}Q^q)\), where \( \Sigma^{-n-1}D(W) \cong L \) and \( q \) induces a map \( W \to \Sigma^{-n-1}D(W) \cong L \) having cofiber \( V \). Before proving Theorem 1, we are free to replace \( L \) by a cobordant Lagrangian by doing surgery on the quadratic object \( W \). We may therefore assume that \( W \) has been simplified by means of (algebraic) surgery below the middle dimension. Write \( n = 2k \) or \( n = 2k + 1 \). We may assume that \( W \) is \((-k-1)\)-connective, so that \( \Sigma^k W \cong \Sigma^{-1}D(W) \) has projective amplitude \( \le k \). Note in particular that \( \Sigma^k W \) is connected, and \( \Sigma^k V \) is connected (since \( H_0(\tilde{M}; \mathbb{Z}) \cong H_0(\tilde{X}; \mathbb{Z}) \cong \mathbb{Z} \)), so that \( \Sigma^k L \) is connected.

We now observe that the following conditions are equivalent for an integer \( 1 \le p \le \frac{n}{2} \):

\begin{enumerate}
  \item The map \( f : M \to X \) is \( p \)-connected.
  \item The map \( \tilde{f} : \tilde{M} \to \tilde{X} \) is \( p \)-connected.
  \item The spectrum \( \Sigma^k V \) is \( p \)-connective.
  \item The spectrum \( \Sigma^k L \) is \( p \)-connective.
\end{enumerate}

The equivalence of \((a) \) and \((b) \) follows from the fact that \( f \) and \( \tilde{f} \) have the same homotopy fibers. The equivalence of \((b) \) and \((c) \) follows from the homotopy equivalence \( C_\ast(\tilde{M}; \mathbb{Z}) \cong C_\ast(\tilde{X}; \mathbb{Z}) \oplus \Sigma^k V \). To prove that \((c) \Rightarrow (d) \), we note that there is a fiber sequence

\[
\Sigma^k L \to \Sigma^k V \to D L.
\]

The homotopy groups \( \pi_i D(L) \cong \pi_i \Sigma^{n+1}(W) \) vanish for \( i < \frac{n}{2} \), so that \( \pi_i \Sigma^n L \to \pi_i \Sigma^n V \) is bijective for \( i < \frac{n}{2} \). This proves \((c) \Leftrightarrow (d) \).

Suppose that there exists an integer \( p < k - 1 \) such that \( \pi_p \Sigma^n L \ne 0 \). Choose \( p \) as small as possible, so that \( \Sigma^n L \) is \( p \)-connective. Any choice of element in \( \pi_p \Sigma^n L = \pi_p \Sigma^n L \) determines a map \( \Sigma^{p-n} Z[G] \to L \).
Composing with the map $L \to V$, we obtain a map $u : \Sigma^p-nZ[\mathbb{Z}] \rightarrow V$ and a nullhomotopy of $q|\Sigma^p-nZ[\mathbb{Z}]$. According to Theorem 4, we can lift this data to normal surgery datum $\alpha : S^p \times D^m \ni \sigma \mapsto M$. Let $f' : M' \rightarrow X$ be the normal map obtained by surgery along $\alpha$, and let $(V', q')$ be the corresponding representative for $\sigma$. Then $(V', q')$ is obtained from (algebraic) surgery on $V$ along $u$. It follows that $L$ determines a Lagrangian $L'$ in $V'$, where $L'$ is the cofiber of the map $\Sigma^p-nZ[\mathbb{Z}] \rightarrow L$. Since $L$ is finitely presented as a $\mathbb{Z}[\mathbb{G}]$-module spectrum, its bottom homotopy group is finitely generated as a discrete $\mathbb{Z}[\mathbb{G}]$-module. Consequently, after finitely many application of this procedure, we can reduce to the case where $\pi_p \Sigma^n L \simeq 0$: that is, where $\Sigma^n L$ is $p + 1$-connective.

Applying the above argument finitely many times, we may reduce to the case where $\pi_p \Sigma^n L \simeq \pi_{p-n} L$ vanishes for $p < k - 1$. Consequently, we see that $\Sigma^n L$ is $(k-1)$-connective and has projective amplitude $\leq k$. Since $L$ is finitely presented, we can argue as in the proof of the $\pi$-$\pi$ theorem to deduce that there is a fiber sequence

$$\Sigma^{k-1} \mathbb{Z}[\mathbb{G}] \overset{\phi}{\rightarrow} \Sigma^n L \rightarrow (\Sigma^k \mathbb{Z}[\mathbb{G}])^{m'}.$$  

for some integers $m$ and $m'$. If $m > 0$, then the restriction of $\phi$ to a summand of $(\Sigma^{k-1} \mathbb{Z}[\mathbb{G}])^m$ yields a map $\Sigma^p-n \mathbb{Z}[\mathbb{G}] \rightarrow L$, where $p = k - 1$. We therefore obtain a composite map $\Sigma^p-n \mathbb{Z}[\mathbb{G}] \rightarrow L \rightarrow V$ and a nullhomotopy of $q|\Sigma^p-n \mathbb{Z}[\mathbb{G}]$. Invoking Theorem 4, we can lift this to a normal surgery datum. Performing surgery along this datum (and replacing $M$ by the result), we can reduce to the case where there is a fiber sequence

$$\Sigma^{k-1} \mathbb{Z}[\mathbb{G}] \overset{\phi}{\rightarrow} \Sigma^n L \rightarrow (\Sigma^k \mathbb{Z}[\mathbb{G}])^{m'}.$$  

Applying this procedure finitely many times, we reduce to the case $m = 0$: that is, $\Sigma^n L \simeq (\Sigma^k \mathbb{Z}[\mathbb{G}])^{m'}$. If $m' > 0$, we can restrict the map $L \rightarrow V$ to a summand of $L$ to obtain a map $\Sigma^{k-n} \mathbb{Z}[\mathbb{G}] \rightarrow V$ and a nullhomotopy of $q|\Sigma^{k-n} \mathbb{Z}[\mathbb{G}]$. Invoking Theorem 4 again, we can perform surgery to reduce to the case $\Sigma^n L \simeq (\Sigma^k \mathbb{Z}[\mathbb{G}])^{m'-1}$. Applying this procedure finitely many times, we reduce to the case $L \simeq 0$. The fiber sequence

$$L \rightarrow V \rightarrow \Sigma^{-n}DL$$

shows that $V \simeq 0$, so that the map $\tilde{f} : \tilde{M} \rightarrow \tilde{X}$ induces an isomorphism on homology. Since $\tilde{M}$ and $\tilde{X}$ are simply connected, we deduce that $\tilde{f}$ is a homotopy equivalence, so that $f : M \rightarrow X$ is also a homotopy equivalence. This completes the proof of Theorem 1 (modulo Theorem 4).