Surgery (Lecture 32)

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Our goal today is to begin the proof of the following:

**Theorem 1.** Let $X$ be a Poincare pair of dimension $n \geq 5$, $\zeta$ a stable PL bundle on $X$, and $f : M \to X$ a degree one normal map, where $M$ is a PL manifold. Let $\sigma_f^\partial \in \Omega^\infty L^\partial q(X, \zeta_X)$ be the relative signature of $f$, and suppose we are given a path $p$ from $\sigma_f^\partial$ to the base point of $\Omega^\infty L^\partial q(X, \zeta_X)$. (We can identify such a path with a Lagrangian in the Poincare object representing $\sigma_f^\partial$, which is well-defined up to bordism.) Then there exists a $\Delta^1$-family of degree one normal maps $F : B \to X \times \Delta^1$, where $B$ is a bordism from $M = F^{-1}(X \times \{0\})$ to a PL manifold $N = F^{-1}(X \times \{1\})$ such that $F$ induces a homotopy equivalence $f' : N \to X$. Moreover, we can arrange that $F$ determines a path from $\sigma_f^\partial$ to $\sigma_f^{\partial'} = 0$ which is homotopic to $p$.

**Remark 2.** In the last lecture, we sketched the formulation of a more general version of Theorem 1, where we replace $f$ by a Poincare pair $(X, \partial X)$ where $\partial X$ is already a PL manifold. To simplify the discussion, we will restrict our attention to the case where $\partial X = \emptyset$, but the ideas introduced in this lecture generalize to the relative case.

To prove Theorem 1, we need a method for producing bordisms between PL manifolds. For this, we will use the method of surgery. Fix a PL manifold $M$ of dimension $n$. Write $n = p + q + 1$. Let $D^{p+1}$ and $D^{q+1}$ denote PL disks of dimension $p + 1$ and $q + 1$, respectively. Let $S^p$ and $S^q$ denote their boundaries (spheres of dimension $p$ and $q$, respectively).

**Definition 3.** A $p$-surgery datum on $M$ is a PL embedding $\alpha : S^p \times D^{q+1} \to M$.

To a first approximation, a $p$-surgery datum $\alpha$ on $M$ is given by an embedding of PL manifolds $\alpha_0 : S^p \to M$ (given by restricting $\alpha$ to the product of $S^p$ with the center of $D^{q+1}$). To obtain a surgery datum from $\alpha_0$, we must additionally specify that $\alpha_0$ extends to a PL homeomorphism between $S^p \times D^{q+1}$ and a neighborhood of the image of $\alpha_0$. Such a homeomorphism determines a smooth structure on $M$ along the image of $\alpha_0$, with respect to which $\alpha_0$ is a smooth embedding with trivialized normal bundle. Conversely, suppose we are given an embedding $\alpha_0 : S^p \to M$ and a smoothing of $M$ along the image of $\alpha_0$, such that $\alpha_0$ is a smooth map. Then $\alpha_0$ has a normal bundle $N_{\alpha_0}$, and there is a neighborhood of $N_{\alpha_0}(S^p)$ in $M$ which is diffeomorphic to the unit sphere bundle of $N_{\alpha_0}$. In particular, if $N_{\alpha_0}$ is trivial, we obtain a diffeomorphism (and therefore a PL homeomorphism) of a neighborhood of $\alpha_0(S^p)$ with $S^p \times D^{q+1}$. This argument shows that we can identify a $p$-surgery datum on $M$ with three pieces of data:

(i) A PL embedding $\alpha_0 : S^p \to M$.

(ii) A smoothing of $M$ along the image of $\alpha_0$ (with respect to which $\alpha_0$ is a smooth map).

(iii) A trivialization of the normal bundle to $\alpha_0$ (as a vector bundle).

**Construction 4.** Let $M$ be a PL manifold of dimension $n = p + q + 1$ and let $\alpha : S^p \times D^{q+1} \to M$ be a $p$-surgery datum. We let $B(\alpha)$ denote the polyhedron given by

$$(M \times [0,1]) \bigsqcup_{\{1\} \times S^p \times D^{q+1}} (D^{p+1} \times D^{q+1}).$$
Then \( B(\alpha) \) is a PL manifold with boundary, given by the disjoint union of \( M \times \{0\} \) and
\[
N = M - (S^p \times (D^{q+1})^\circ) \coprod_{S^p \times S^q} (D^{p+1} \times S^q).
\]

We refer to \( N \) as the PL manifold obtained from \( M \) via surgery along \( \alpha \), and to \( B(\alpha) \) as the trace of the surgery.

More informally: \( N \) is the manifold obtained from \( M \) by removing the interior of \( S^p \times D^{q+1} \) (thereby creating a manifold with boundary \( S^p \times S^q \)) and gluing on a copy of \( D^{p+1} \times S^q \).

**Remark 5.** Let \( N \) be a PL manifold obtained from surgery on a PL manifold \( M \) along a map \( \alpha : S^p \times D^{q+1} \hookrightarrow M \). Then there is an evident embedding \( \beta : D^{p+1} \times S^q \to N \), which is a \( q \)-surgery datum in \( N \). Performing surgery on \( N \) along \( \beta \) recovers the manifold \( M \).

We will be interested in using surgery to construct normal bordisms between normal maps to a Poincare complex. For this, we need a slight variation on Definition 3. Let \( M \) be a PL manifold, so that the stable normal bundle of \( M \) is classified by a map \( \chi : M \to \mathbb{Z} \times \text{BPL} \). If we are given a \( p \)-surgery datum \( \alpha : S^p \times D^{q+1} \to M \), then \( \chi \circ \alpha \) extends canonically to a map \( \gamma : D^{p+1} \times D^{q+1} \to \mathbb{Z} \times \text{BPL} \).

Suppose now that \( X \) is a space equipped with a stable PL bundle \( \zeta \), and that we are given a normal map \( f : M \to X \). Then \( \zeta \) is classified by a map \( \chi_X : X \to \mathbb{Z} \times \text{BPL} \), and the normal structure on \( f \) gives a homotopy \( h_0 : \chi \simeq \chi_X \circ f \).

**Definition 6.** In the situation above, a normal \( p \)-surgery datum on \( M \) consists of the following data:

(i) A \( p \)-surgery datum \( \alpha : S^p \times D^{q+1} \to M \).

(ii) A map \( \beta : D^{p+1} \times D^{q+1} \to X \) extending \( f \circ \alpha \).

(iii) A homotopy \( h \) from \( \chi_X \circ \beta \) to \( \gamma \), extending the homotopy determined by \( h \).

Given a normal \( p \)-surgery datum, we can use \( \alpha \) to construct a bordism \( B(\alpha) \) from \( M \) to a PL manifold \( N \), \( \beta \) to construct a map \( F : B(\alpha) \to X \) extending \( f : M \to X \), and \( h \) to endow \( F \) with the structure of a \( \Delta^1 \)-family of normal maps.

**Remark 7.** Let us think of a \( p \)-surgery datum on a PL manifold \( M \) as an embedding \( \alpha_0 : S^p \to M \), together with a choice of trivial normal bundle to \( \alpha_0 \). If \( f : M \to X \) is a degree one normal map, then to obtain a normal \( p \)-surgery datum we need to choose a nullhomotopy of the composite map \( (f \circ \alpha_0) : S^p \to X \), which is compatible with the nullhomotopy of the map
\[
S^p \xrightarrow{\alpha_0} M \xrightarrow{f} X \to \mathbb{Z} \times \text{BPL}
\]
determined by the choice of trivial normal bundle.

Let us now see what surgery can do for us in low degrees. Assume that \( X \) is a Poincare space of dimension \( n \geq 5 \), \( \zeta \) a stable PL bundle on \( X \), and \( f : M \to X \) is a degree one normal map.

Let us begin by doing surgery in the case \( p = -1 \). In this case, \( S^p \) is empty and therefore a surgery datum \( \alpha : S^p \times D^{q+1} \to M \) is unique. To promote \( \alpha \) to a normal surgery datum, we need to choose a map \( \beta : D^{n+1} \to X \) (up to homotopy, this a point \( x \in X \)), together with a trivialization of \( \beta^* \zeta \). Unwinding the definitions, we see that \( B(\alpha) \) is the disjoint union \( (M \times [0,1]) \coprod D^{n+1} \), regarded as a bordism from \( M \) to \( M \coprod S^n \). If we have chosen \( \beta \) and the trivialization of \( \beta^* \zeta \), then we can regard this as a normal bordism from \( f \) to a map \( M \coprod S^n \to X \), whose restriction to \( S^n \) is determined by \( \beta \). By performing surgeries of this type, we can always arrange that the map \( M \to X \) is surjective on connected components.

Now suppose that \( f : M \to X \) fails to be injective on connected components. Then we can choose two points \( x, y \in M \) belonging to different components of \( M \) and a path joining \( f(x) \) to \( f(y) \). Choosing small
disks around the points $x$ and $y$, we obtain a 0-surgery datum $\alpha : S^0 \times D^n \hookrightarrow M$. A choice of path $p$ from $f(x)$ to $f(y)$ determines the datum (ii) required by Definition 6. We cannot always extend $\alpha$ to a normal surgery datum: our choice of disks determines trivializations of the fibers $\xi_{f(x)}$ and $\xi_{f(y)}$, which may or may not extend to a trivialization of $\zeta$ over the path $p$. However, the obstruction is slight by virtue of the following (non-obvious!) fact:

Claim 8. The fundamental group $\pi_1(\mathbb{Z} \times \text{BPL})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In other words, every orientation-preserving PL automorphism of $\mathbb{R}^n$ is isotopic to the identity, for $n \geq 0$.

In fact, more is true: the map $\pi_i(\mathbb{Z} \times \text{BO}) \to \pi_i(\mathbb{Z} \times \text{BPL})$ induces an isomorphism for $i \leq 6$ and a surjection when $i = 7$ (using smoothing theory, this is equivalent to the assertion that there are no exotic smooth structures on piecewise linear spheres of dimensions $\leq 6$). In this lecture, we will need something much weaker: namely, that the above map is bijective for $i \leq 1$ and surjective for $i \leq 2$. Using smoothing theory, this is equivalent to the (reasonably obvious) claim that there are no exotic smooth structures on spheres of dimension $\leq 1$.

In our situation, we cannot necessarily extend an arbitrary $\alpha : S^0 \times D^n \hookrightarrow M$ to a normal surgery datum. However, we always do so after modifying $\alpha$ by applying an orientation-reversing automorphism to one of the disks $D^n$. After making this modification, we obtain a normal bordism from $M$ to a PL manifold with fewer connected components. Applying this procedure finitely many times, we may replace $f : M \to X$ by a degree one normal map which induces an isomorphism $\pi_0 M \to \pi_0 X$.

Let us now assume that $X$ and $M$ are connected, and choose a base point $x \in M$. Suppose that the map $\pi_1 M \to \pi_1 X$ is not surjective. Choose another point $y \in M$ and a path $q$ from $y$ to $x$. Choose any class $\gamma$ in $\pi_1 X$, and a path $p$ from $f(x)$ to $f(y)$ such that the loop composing $p$ with $f(q)$ represents $\gamma$. Choosing small disks around $x$ and $y$, we obtain a surgery datum $\alpha : S^0 \times D^n \hookrightarrow M$ as before. The path $p$ supplies the datum (ii) required by Definition 6, and we can argue as before (modifying $\alpha$ if necessary) to obtain the datum (iii). Let $N$ be obtained from $M$ by normal surgery along $\alpha$. Since $n \geq 3$, deleting small disks around $x$ and $y$ does not change the fundamental group of $M$. Using van Kampen’s theorem, we compute that $\pi_1 N$ is obtained from $\pi_1 M$ by freely adjoining an additional generator, and the map $\pi_1 N \to \pi_1 X$ carries this generator to $\gamma$ (here we are being sloppy about base points here). Since $X$ is a finite complex, its fundamental group is finitely generated. We may therefore perform this procedure finitely many times to reduce to the situation where the degree one normal map $f : M \to X$ induces a surjection $\pi_1 M \to \pi_1 X$.

Now suppose that $\pi_1 M \to \pi_1 X$ fails to be injective. Choose an element of $\pi_1 M$ whose image in $\pi_1 X$ is trivial. We can represent this element by a map $\alpha_0 : S^1 \to M$. Since the dimension of $M$ is $\geq 3$, a general position argument allows us to assume that $\alpha_0$ is an embedding. The composite map $S^1 \to M \to X$ is nullhomotopic, so that the stable normal bundle of $M$ is trivial in a neighborhood of $\alpha_0$ and we may therefore assume that $M$ is smooth in a neighborhood of $\alpha_0$. The normal bundle to $\alpha_0$ is stable trivial, hence orientable and therefore trivial. We may therefore extend $\alpha_0$ to an embedding $\alpha : S^1 \times D^{n-1} \hookrightarrow M$. Choose a nullhomotopy of $f \circ \alpha$. As before, it is not clear that we can choose datum (iii) required by Definition 6: we encounter an obstruction in $\pi_2(\mathbb{Z} \times \text{BPL})$. However, since the map $\pi_2(\mathbb{Z} \times \text{BO}) \to \pi_2(\mathbb{Z} \times \text{BPL})$ is surjective, we can adjust our original embedding $\alpha$ (choosing a different trivialization of the normal bundle to $\alpha_0$) to make this obstruction vanish. This allows us to perform a normal surgery on the manifold $M$, thereby obtaining a cobordant degree one normal map $f' : N \to X$. Since the dimension of $M$ is $\geq 4$, removing a neighborhood of $\alpha_0(S^1)$ does not change the fundamental group of $M$. Consequently, we can use van Kampen’s theorem to compute the fundamental group of $N$: it is obtained from the fundamental group of $M$ by killing the normal subgroup generated by $\gamma$.

Since $X$ is a finite complex, the fundamental group $\pi_1 X$ is finitely presented. Since $\pi_1 M$ is finitely generated, the surjective map $\pi_1 M \to \pi_1 X$ exhibits $\pi_1 X$ as the quotient of $\pi_1 M$ by the normal subgroup generated by finitely many elements of $\pi_1 M$. It follows that, after a finite number of applications of the above procedure, we may replace $f : M \to X$ by a degree one normal map which induces an isomorphism of fundamental groups.