Poincare Spaces and Spivak Fibrations (Lecture 26)

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Let $X$ be a topological space and $\mathcal{C}$ an $\infty$-category. We let $\text{Shv}_{lc}(X; \mathcal{C})$ denote the $\infty$-category of maps from the Kan complex $\text{Sing}_\bullet(X)$ into $\mathcal{C}$. We will refer to $\text{Shv}_{lc}(X; \mathcal{C})$ as the $\infty$-category of locally constant $\mathcal{C}$-valued sheaves on $X$, or sometimes as the $\infty$-category of local systems of $\mathcal{C}$-valued sheaves on $X$. If $X$ is a polyhedron with triangulation $T$, then we can identify $\text{Shv}_{lc}(X; \mathcal{C})$ with the full subcategory of $\text{Shv}_{\tau}(X; \mathcal{C})$ spanned by those functors which carry each inclusion $\tau \subseteq \tau'$ of simplices to an invertible morphism of $\mathcal{C}$.

Let $f : X \rightarrow Y$ be a map of topological spaces. Then $f$ induces a pullback functor $f^* : \text{Shv}_{lc}(Y; \mathcal{C}) \rightarrow \text{Shv}_{lc}(X; \mathcal{C})$. Suppose that $\mathcal{C}$ is the $\infty$-category of spectra. Then $f^*$ preserves all limits and colimits, and therefore admits both a left adjoint $f_!$ and a right adjoint $f_*$. In the special case where $Y$ is a point, we will denote the functors $f_!$ and $f_*$ by $C_*(X; \bullet)$ and $C^*(X; \bullet)$, respectively. If $X$ is a polyhedron with triangulation $T$, these are described by the formulas

$$C_*(X; \mathcal{F}) = \lim_{\tau} \mathcal{F}(\tau), \quad C^*(X; \mathcal{F}) = \lim_{\tau} \mathcal{F}(\tau).$$

If $X$ is a finite polyhedron, we conclude that the construction $C^* : \text{Shv}_{lc}(X; \text{Sp}) \rightarrow \text{Sp}$ commutes with homotopy colimits.

Suppose now that $X$ is connected with base point $x$. Then $\text{Shv}_{lc}(X; \text{Sp})$ can be identified with the $\infty$-category of modules over the $A_{\infty}$-ring $R = \Sigma_\infty \Omega(X)$. Any functor $F : \text{LMod}_R \rightarrow \text{Sp}$ is determined by its value $F(R) \in \text{Sp}$, together with its right $R$-module structure. Indeed, the fact that $F$ commutes with homotopy colimits implies that $F$ is given by $F(M) \simeq F(R) \wedge_R M$. We can identify $F(R)$ with a local system $\zeta$ on $X$, so that $F$ is given by the formula $F(M) = C_*(X; M \wedge \zeta)$. This description generalizes immediately to the case where $X$ is not assumed to be connected:

**Proposition 1.** Let $F : \text{Shv}_{lc}(X; \text{Sp}) \rightarrow \text{Sp}$ be a functor which commutes with homotopy colimits. Then $F$ is given by $F(\mathcal{F}) = C_*(\mathcal{F} \wedge \zeta)$, where $\zeta$ is a local system of spectra on $X$. Moreover, the local system $\zeta$ is determined uniquely up to equivalence.

In particular, if $X$ is a finite polyhedron (or any space equivalent to a finite polyhedron) and $f : X \rightarrow *$ denotes the projection map, we have an equivalence of functors

$$f_*(\bullet) \simeq C_*(\bullet \wedge \zeta_X)$$

for some local system $\zeta_X$ on $X$.

**Definition 2.** We say that a finite polyhedron $X$ is a Poincare space if $\zeta_X$ is a spherical fibration (that is, if each of the fibers $\zeta_X(x)$ is an invertible spectrum). In this case, we say that $\zeta_X$ is the Spivak normal fibration of $X$.

**Remark 3.** Let $X$ be a finite polyhedron containing a point $x$. Let $i : \{x\} \rightarrow X$ denote the inclusion and $p : X \rightarrow *$ the projection map, so that $p \circ i$ is a homeomorphism. Then we have a homotopy equivalence of spectra

$$i^* \zeta_X \simeq (p \circ i)_! i^* \zeta_X \simeq p_!(i_! i^* \zeta_X) \simeq p_!(i_! S \wedge \zeta_X) \simeq C^*(X; i_! S).$$
In other words, the stalk \( \zeta_X(x) \) is given by taking global sections of the local system of spectra on \( X \) that assigns to each point \( y \in X \) the suspension spectra \( \Sigma^\infty P_{x,y} \), where \( P_{x,y} \) denotes the path space \( \{ p : [0,1] \to X : p(0) = x, p(1) = y \} \).

**Remark 4.** Let \( \mathcal{S} \) denote the constant local system on \( X \) with value the sphere spectrum, so we have a canonical map \( S \to C^*(X; \mathcal{S}) \simeq C_*(X; \mathcal{S} \wedge \zeta_X) \simeq C_*(X; \zeta_X) \). We can identify this map with a point of \( \Omega^\infty C_*(X; \zeta_X) \), which we will refer to as the fundamental class of \( X \) and denote by \([X]\).

The fundamental class determines the equivalence of functors \( C^*(X; \bullet) \simeq C_*(X; \bullet \wedge \zeta_X) \): it is given by

\[
C^*(X; \mathcal{F}) \simeq \text{Mor}(\mathcal{S}, \mathcal{F}) \to \text{Mor}(\zeta_X, \mathcal{F} \wedge \zeta_X) \to \text{Mor}(C_*(X; \zeta_X), C_*(X; \mathcal{F} \wedge \zeta_X)) \xrightarrow{[X]} C_*(X; \mathcal{F} \wedge \zeta_X).
\]

**Example 5.** Let \( X \) be a simply connected finite polyhedron. Then \( X \) is a Poincare space if and only if there exists a fundamental class \( \eta_X \in H_n(X; \mathbb{Z}) \) which induces cap product isomorphisms \( \phi_i : H^i(X; \mathbb{Z}) \to H_{n-i}(X; \mathbb{Z}) \). The “only if” direction is obvious: if \( X \) is a Poincare space, then \( \zeta_X \wedge \mathbb{Z} \) is necessarily equivalent to \( \Sigma^{-n} \mathbb{Z} \) (orientability is obvious, since \( X \) is simply connected) so we can take \( \eta_X \) to be the image of the fundamental class \([X]\); the desired result then follows from the equivalence

\[
C^*(X; \mathbb{Z}) \simeq C_*(X; \mathbb{Z} \wedge \zeta_X) \simeq C_*(X; \Sigma^{-n} \mathbb{Z}).
\]

The converse requires the simple connectivity of \( X \). Note that \( \eta_X \) induces a map of spectra \( C^*(X; \mathbb{Z}) \to \Sigma^{-n} \mathbb{Z} \), hence a map \( C_*(X; \mathbb{Z} \wedge \zeta_X) \to \Sigma^{-n} \mathbb{Z} \) which is adjoint to a map \( \theta : \mathbb{Z} \wedge \zeta_X \to \Sigma^{-n} \mathbb{Z} \). We claim that \( \theta \) is invertible (from which it will follow that each fiber of \( \zeta_X \) is equivalent to the invertible spectrum \( \Sigma^{-n} S \)). Since \( X \) is simply connected (and the fibers of \( \zeta_X \) are \( k \)-connective for \( k \ll 0 \)), it will suffice to show that \( \theta \) induces an equivalence after applying the functor \( C_* \). That is, we must show that the canonical map

\[
C^*(X; \mathbb{Z}) \simeq C_*(X; \mathbb{Z} \wedge \zeta_X) \to \Sigma^{-n} C_*(\mathbb{Z})
\]

is a homotopy equivalence. On the level of homotopy groups, this is precisely the condition that the maps \( \phi_i \) are isomorphisms.

Let us now depart from our previous convention and regard quadratic functors as covariant functors from a stable \( \infty \)-category \( \mathcal{C} \) to spectra. If \( R \) is an \( A_\infty \)-ring with involution, we have a quadratic functor \( Q^\mathcal{F} : \text{RMod}_R \to \text{Sp} \) given by

\[
Q^\mathcal{F}(M) = (M \wedge_R M)^{h\mathbb{S}_2},
\]

which restricts to a nondegenerate quadratic functor on \( \text{RMod}_R^{fp} \). If \( X \) is a space equipped with a spherical fibration \( \zeta \) and \( f : X \to * \) denotes the projection map, then we obtain a quadratic functor \( Q^\mathcal{F}_\zeta : \text{Shv}_{lc}(X; \text{RMod}_R) \to \text{Sp} \) given by the formula \( Q^\mathcal{F}_\zeta(\mathcal{F}) = C_*(X; \zeta \wedge Q^\mathcal{F}(\mathcal{F})) \), which is nondegenerate when restricted to the \( \infty \)-category of compact objects of \( \text{Shv}_{lc}(X; \text{RMod}_R) \).

Let \( \mathcal{R} \) denote the constant sheaf on \( X \) having the value \( R \). Given a map of spectra \( \eta : S \to C_*(X; \zeta) \), we obtain a map \( S \to C_*(X; \zeta \wedge R^{h\mathbb{S}_2}) \simeq Q^\mathcal{F}_\zeta(\mathcal{R}) \), which we will denote by \( q \). Then the pair \((\mathcal{R}, q)\) is a quadratic object of \( \text{Shv}_{lc}(X; \text{RMod}_R) \). Let \( B^\mathcal{F}_\zeta \) denote the polarization of \( Q^\mathcal{F}_\zeta \), given by the formula

\[
B^\mathcal{F}_\zeta(\mathcal{F}, \mathcal{F}') = C_*(X; \mathcal{F} \wedge_R \mathcal{F}' \wedge \zeta).
\]

If \( X \) is a Poincare space and \( \zeta = \zeta_X \) is its Spivak normal fibration, then we have a homotopy equivalence

\[
B^\mathcal{F}_\zeta(\mathcal{R}, \mathcal{F}) = C_*(X; \mathcal{R} \wedge_R \mathcal{F} \wedge \zeta) \simeq C_*(X; \mathcal{F} \wedge \zeta) \simeq C^*(X; \mathcal{F}) \simeq \text{Mor}(\mathcal{R}, \mathcal{F}).
\]

This tells us that \( \mathcal{R} \) is self-dual: that is, \((\mathcal{R}, q)\) is a Poincare object of \( \text{Shv}_{lc}(X; \text{RMod}_R) \). We therefore obtain an element \( \sigma^\mathcal{F}_\zeta \in \Omega^\infty L^{vs}(X, \zeta_X, R) \), called the visible symmetric signature of the Poincare complex \( X \).

For later use, we will need a slight generalization of the notion of a Poincare complex. Suppose we are given a map of finite spaces \( \partial X \to X \) (which, up to homotopy equivalence, we may as well suppose is given
by an inclusion between finite polyhedra). Given a local system of spectra \( \mathcal{F} \) on \( X \), we let \( \mathcal{F} \mid \partial X \) denote the pullback of \( \mathcal{F} \) to \( \partial X \), and form fiber sequences

\[
C_*(\partial X; \mathcal{F} \mid \partial X) \to C_*(X; \mathcal{F}) \to C_*(X, \partial X; \mathcal{F})
\]

\[
C^*(X, \partial X; \mathcal{F}) \to C^*(X; \mathcal{F}) \to C^*(\partial X; \mathcal{F} \mid \partial X).
\]

Arguing as above, we see that \( C^*(X, \partial X; \bullet) \) commutes with homotopy colimits and is therefore given by \( \mathcal{F} \mapsto C_*(X; \zeta_{(X, \partial X)} \wedge \mathcal{F}) \) for some local system \( \zeta_{(X, \partial X)} \). The equivalence between \( C^*(X, \partial X; \bullet) \) is determined by a fundamental class \( [X] : S \to C_*(X, \partial X; \zeta_{(X, \partial X)}) \). Note that \([X]\) determines a composite map

\[
[S] : S \to C_*(X, \partial X; \zeta_{(X, \partial X)}) \to \Sigma C_*(X; \zeta_{(X, \partial X)} \mid \partial X) \cong C_*(\partial X; \Sigma \zeta_{(X, \partial X)} \mid \partial X).
\]

**Definition 6.** A pair of finite spaces \((X, \partial X)\) is a Poincare pair if the following conditions are satisfied:

1. The local system \( \zeta_{(X, \partial X)} \) defined above is a spherical fibration.
2. The map \([\partial X]\) is a fundamental class for \( \partial X \): that is, it induces a homotopy equivalence

\[
C^*(\partial X; \mathcal{F}) \to C_*(\partial X; (\Sigma \zeta_{(X, \partial X)} \mid \partial X) \wedge \mathcal{F})
\]

for every local system \( \mathcal{F} \) on \( \partial X \). (So that the Spivak normal fibration of \( \partial X \) is given by \( \Sigma \zeta_{(X, \partial X)} \mid \partial X \).)

**Remark 7.** Let \((X, \partial X)\) be a pair of finite spaces \( \mathcal{F} \) be a local system of spectra on \( X \). We have a commutative diagram of fiber sequences

\[
\begin{array}{ccc}
C^*(X, \partial X; \mathcal{F}) & \to & C^*(X; \mathcal{F}) \\
\downarrow & & \downarrow \\
C_*(X; \zeta_{(X, \partial X)} \wedge \mathcal{F}) & \to & C_*(X, \partial X; \zeta_{(X, \partial X)} \wedge \mathcal{F}) \to C_*(X; (\Sigma \zeta_{(X, \partial X)} \mid \partial X) \wedge \mathcal{F})
\end{array}
\]

where the vertical maps are given by cap product with \([X]\) and \([\partial X]\). The left vertical map is a homotopy equivalence by construction, and the right vertical map is a homotopy equivalence when \((X, \partial X)\) is a Poincare pair. It follows that if \((X, \partial X)\) is a Poincare pair, then the middle map is also a homotopy equivalence: that is, the cap product map

\[
C^*(X; \mathcal{F}) \to C_*(X, \partial X; \zeta_{(X, \partial X)} \wedge \mathcal{F})
\]

is a homotopy equivalence.

Suppose that \( i : \partial X \to X \), and let \( R \) be an \( \mathbb{A}_\infty \)-ring with involution. We have a visible symmetric signature \( \sigma^\wedge_{\partial X} \in L^\wedge(\partial X, \zeta_{\partial X}, R) \), given by \((\mathcal{R}, q)\). Then \( q \) determines a symmetric bilinear form \( q_\theta \) on the object \( i^*R \in \text{Shv}_{\mathbf{lc}}(X; \text{RMod}_R) \) with respect to the quadratic functor \( Q_{\Sigma \zeta_{(X, \partial X)}} \). We have a canonical map \( i^*R \to R \), and a fiber sequence

\[
C_*(X, \partial X; \zeta_{(X, \partial X)} \wedge \mathcal{R}^{h\Sigma_{\zeta_{(X, \partial X)}}}) \to C_*(X; \Sigma \zeta_{(X, \partial X)} \wedge i^*R^{h\Sigma_{\zeta_{(X, \partial X)}}}) \to C_*(X; \Sigma \zeta_{(X, \partial X)} \wedge \mathcal{R}^{h\Sigma_{\zeta_{(X, \partial X)}}}).
\]

Consequently, the fundamental class \([X]\) provides a nullhomotopy of the image of \( q_\theta \) in \( Q_{\Sigma \zeta_{(X, \partial X)}}(R) \). This nullhomotopy exhibits \( R \) as a Lagrangian for the Poincare object \((i^*R; q_\theta)\). In other words, it gives a canonical lifting of \( \sigma^\wedge_{\partial X} \) to the homotopy fiber of the map

\[
L^\wedge(\partial X, \zeta_{\partial X}, R) \to L^\wedge(X, \Sigma \zeta_{(X, \partial X)}, R).
\]

Let us denote this lifting by \( \sigma^\wedge_{X} \). We will refer to it as the visible symmetric signature of \( X \) (or the visible symmetric signature of the Poincare pair \((X, \partial X)\)).
**Notation 8.** Let $f : Y \to X$ be a map of spaces and let $\zeta$ be a spherical fibration on $X$. We let $L^v s(X, Y, \zeta, R)$ denote the homotopy cofiber of the map

$$L^v s(Y, f^* \zeta, R) \to L^v s(X, \zeta, R).$$

Equivalently $L^v s(X, Y, \zeta, R)$ is the homotopy fiber of the map

$$L^v s(Y, f^* \Sigma \zeta, R) \to L^v s(X, \Sigma \zeta, R).$$

The upshot of the above discussion is that if $(X, \partial X)$ is a Poincare pair, we can identify $\sigma^v s_X$ with a point in the 0th space of $L^v s(X, \partial X, \zeta_{(X, \partial X)}, R)$. When $\partial X = \emptyset$, this specializes to the definition of the visible symmetric signature of a Poincare space described earlier.