In the last lecture, we introduced the \( L^\bullet(X, \zeta, R) \) and \( L^\circ(X, \zeta, R) \), where \( R \) is an \( A_\infty \)-ring with involution, \( X \) is a finite polyhedron, and \( \zeta \) is a spherical fibration on \( X \). When \( \zeta \) is trivial, these spectra are given simply by \( X \wedge L^\bullet(R) \) and \( X \wedge L^\circ(R) \), respectively. In general, they depend on the spherical fibration \( \zeta \). However, our excision argument generalizes to show that \( L^\bullet(X, \zeta, R) \) is given by the homotopy colimit
\[
\lim_{\tau \in T} L(LMod_{fp}^R, \zeta(\tau) \wedge Q^0)
\]
where \( T \) denotes any triangulation of \( X \). In other words, the homotopy groups of \( L^\bullet(X, \zeta, R) \) are given by the homology of \( X \) with coefficients in a local system of spectra, given by \( (x \in X) \mapsto L(LMod_{fp}^R, \zeta(x) \wedge Q^0) \).

This raises the following general question:

**Question 1.** Let \( \mathcal{C} \) be a stable \( \infty \)-category equipped with a nondegenerate functor \( Q \), and let \( E \) be an invertible spectrum. What is the relationship between the \( L \)-theory spectra \( L(\mathcal{C}, Q) \) and \( L(\mathcal{C}, E \wedge Q) \)?

In the situation of Question 1, we can write \( E \simeq \Sigma^{-n} \) for some integer \( n \). We have seen that there is a canonical isomorphism \( L_k(\mathcal{C}, \Omega^n Q) = L_{k+n}(\mathcal{C}, Q) \), suggesting that we should have an equivalence of \( L \)-theory spectra \( L(\mathcal{C}, \Omega^n Q) \simeq \Omega^n L(\mathcal{C}, Q) \). In other words, we have a homotopy equivalence
\[
\theta_E : L(\mathcal{C}, E \wedge Q) \simeq E \wedge L(\mathcal{C}, Q).
\]

For our purposes, we need to know this not just for an individual invertible spectrum \( E \), but in the case where \( E \) ranges over the fibers of some spherical fibration. It is therefore important that our analysis be functorial with respect to automorphisms of \( E \). In fact, it is not possible to choose \( \theta_E \) to be functorial with respect to all automorphisms of \( E \). However, we will show that it can be chosen to depend naturally on automorphisms which are of geometric origin.

**Definition 2.** Let \( M \) be PL manifold, and let \( S \) denote the local system of spectra on \( M \) taking the constant value \( S \) (where \( S \) is the sphere spectrum). The Verdier dual \( \mathbb{D}(S) \) is a spherical fibration over \( M \). We will denote the inverse of this spherical fibration by \( \zeta_M \). We refer to \( \zeta_M \) as the normal spherical fibration of \( M \).

Unwinding the definitions, it can be described by the formula
\[
\zeta_M(x) = (\Sigma^\infty (M/M - \{x\}))^{-1}.
\]

There is a canonical map of spectra \( S \to \Gamma(M; \mathbb{S}) \). If \( M \) is compact, this dualizes to give a map
\[
\Gamma(M; \mathbb{D}S) \simeq \mathbb{D}\Gamma(M; S) \to S.
\]

This map gives a point in the zeroth space of the spectrum
\[
\text{Mor}_{Sp}(\lim_{\tau \in T} \mathbb{D}(S)(\tau), S) \simeq \lim_{\tau \in T} \zeta_M(\tau)
\]
where \( T \) denotes some triangulation of \( M \). We will denote this point by \([M]\) and refer to it as the fundamental class of \( M \).
More generally, if \( M \) is a PL manifold with boundary, we let \( \zeta_M \) denote the local system of spectra on \( M \) obtained by extending the normal spherical fibration from the interior of \( M \) (note that the interior of \( M \) is homotopy equivalent to \( M \), so there exists an essentially unique extension). In this case, we have a fundamental class

\[
[M] \in \Omega^\infty(\lim_{\tau \in \mathcal{T}} \zeta_M(\tau) \cup 0 \quad \text{if } \tau \notin \partial M \\
0 \quad \text{otherwise.}
\]

Let us now fix a PL manifold with boundary \( M \). Let \( \mathcal{C} \) be a stable \( \infty \)-category equipped with a nondegenerate quadratic functor \( Q \). For each triangulation \( T \) of \( M \), let

\[
Q_{\zeta_M, T} : \text{Shv}_T(M, \partial M; \mathcal{C})^{\text{op}} \to \text{Sp}
\]

be given by the formula

\[
\lim_{\tau \in \mathcal{T}} \left\{ Q_{\zeta_M, T}(\mathcal{F}(\tau)) \cup \zeta_M(\tau) \quad \text{if } \tau \notin \partial M \\
0 \quad \text{otherwise.}
\right.
\]

Let \( C \in \mathcal{C} \) be an object, and let \( C \) denote the constant sheaf on \( M \) with taking the value \( C \) (which we will identify with its image in \( \text{Shv}_T(M, \partial M; \mathcal{C}) \)). We then obtain a homotopy equivalence

\[
Q_{\zeta_M, T}(C) \simeq \lim_{\tau \in \mathcal{T}} \left\{ \zeta_M(\tau) \quad \text{if } \tau \notin \partial M \\
0 \quad \text{otherwise.}
\right.
\]

In particular, the fundamental class \([M]\) determines a map

\[
Q(C) \to Q_{\zeta_M, T}(C),
\]

which we will denote by \( q \mapsto q[M] \). This construction carries Poincare objects to Poincare objects, and induces a map of \( L \)-theory spectra

\[
\Phi : L(\mathcal{C}, Q) \to L(\text{Shv}_{\text{const}}(M, \partial M; \mathcal{C}); Q_{\zeta_M})
\]

(where \( Q_{\zeta_M} \) denotes the amalgamation of the quadratic functors \( Q_{\zeta_M, T} \) where \( T \) ranges over all triangulations of \( M \)).

**Example 3.** Let \( M \) be a piecewise linear disk. For every point \( x \) in the interior of \( M \), we have a canonical homotopy equivalence of pairs \( (M, \partial M) \to (M, M - \{x\}) \). Consequently, \( \zeta_M \) is canonically equivalent to the constant sheaf taking the value \( E \), where \( E^{-1} = \Sigma^\infty(M/\partial M) \). It follows that \( L(\text{Shv}_{\text{const}}(M, \partial M; \mathcal{C}), Q\zeta) \) is given by \((M, \partial M) \land L(\mathcal{C}, E \land Q) \simeq E^{-1} \land L(\mathcal{C}, E \land Q)\). We may therefore identify \( \Phi \) with a map of spectra \( E \land L(\mathcal{C}, Q) \to L(\mathcal{C}, E \land Q) \).

Suppose \( M \simeq \Delta^n \). Then \( \Phi \) determines maps \( L_{k+n}(\mathcal{C}, Q) \to L_k(\mathcal{C}, \Omega^n Q) \), which can be identified with the shift isomorphisms defined earlier. It follows that \( \Phi \) is a homotopy equivalence whenever \( M \) is a piecewise linear disk.

The construction of \( \Phi \) is functorial with respect to piecewise linear homeomorphisms of the PL manifold \( M \).

Let us now introduce some terminology to describe the situation more systematically.

Let \( X \) be a polyhedron. A **closed \( n \)-disk bundle** over \( X \) is a map of polyhedra \( q : D \to X \) such that every point \( x \in X \) has an open neighborhood \( U \) for which there is a PL homeomorphism \( q^{-1}U \simeq U \times \Delta^n \) (which commutes with the projection to \( U \)).

There is a canonical bijection between isomorphism classes of closed \( n \)-disk bundles over \( X \) and homotopy classes of maps \( X \to B \text{Disk}(n) \), where \( B \text{Disk}(n) \) denotes the classifying space of the (simplicial) group \( \text{Disk}(n) \) of PL homeomorphisms of \( \Delta^n \).

The disjoint union \( \coprod_n B \text{Disk}(n) \) is equipped with a multiplication which is associative up to coherent homotopy, classifying the formation of products of closed \( n \)-disk bundles. We can describe the group completion of \( \coprod_n B \text{Disk}(n) \) as a product \( \mathbb{Z} \times \text{BPL} \), where \( \text{BPL} \) is the direct limit \( \lim_{\tau n} B \text{Disk}(n) \).
Let Pic(S) denote the classifying space for invertible spectra (so that homotopy classes of maps $X \to \text{Pic}(S)$ correspond to equivalence classes of spherical fibrations over $X$). Every closed disk bundle $q : D \to X$ has an associated spherical fibration, given by $x \mapsto \Sigma^\infty(D_x/\partial D_x)$. This construction determines a map $\coprod_n B\text{Disk}(n) \to \text{Pic}(S)$, which is multiplicative up to coherent homotopy and therefore extends to a map $\mathbb{Z} \times \text{BPL} \to \text{Pic}(S)$.

Fix $(\mathcal{E}, Q)$ as above. Over the space Pic(S), we have two local systems of spectra: one given by the formula $E \mapsto L(\mathcal{E}, E \wedge Q)$ and one given by the formula $E \mapsto E \wedge L(\mathcal{E}, Q)$. The above analysis implies that these two local systems are canonically equivalent when restricted to $\mathbb{Z} \times \text{BPL}$. This proves the following:

**Proposition 4.** Let $X$ be a finite polyhedron with triangulation $T$, $\mathcal{E}$ a stable $\infty$-category equipped with a nondegenerate quadratic functor $Q$, and $\zeta$ a spherical fibration on $X$, classified by a map $X \to \text{Pic}(S)$. Suppose that this classifying map factors through $\mathbb{Z} \times \text{BPL}$ (that is, that the spherical fibration $\zeta$ arises from a closed disk bundle, at least stably). Then there is a homotopy equivalence (depending canonically on the factorization)

$$L(\text{Shv}_{\text{const}}(X; \mathcal{E}), Q_\zeta) \simeq \lim_{\tau \in T} \zeta(\tau) \wedge L(\mathcal{E}, Q).$$

We can also make the analysis of the preceding discussion read in a different way. Let us suppose that $\mathcal{E} = \mathcal{D}^{\text{fp}}(\mathbb{Z})$ is the $\infty$-category of perfect complexes of $\mathbb{Z}$-modules, and let $Q$ be either $Q^a$ or $Q^s$. Then $Q$ is a spectrum valued functor which factors through the $\infty$-category of $\mathbb{Z}$-module spectra. It follows that for every spectrum $E$, we can write $E \wedge Q \simeq (E \wedge \mathbb{Z}) \wedge Q$, so that $E \wedge Q$ depends only on the generalized Eilenberg-MacLane spectrum $E \wedge \mathbb{Z}$. Let $\zeta$ be a spherical fibration on a polyhedron $X$, and suppose that $\zeta$ assigns to each point $x \in X$ a spectrum $\zeta(x)$ which is homotopy equivalent to $\Sigma^n S$. Suppose further that $\zeta$ is orientable. A choice of orientation determines a canonical homotopy equivalence of each $\zeta(x) \wedge \mathbb{Z}$ with $\Sigma^n \mathbb{Z}$, and therefore a natural isomorphism $Q_\zeta \simeq \Sigma^n Q$. It follows that we obtain a canonical homotopy equivalence

$$\lim_{\tau \in T} \zeta(\tau) \wedge L(\mathcal{E}, Q) \simeq L(\text{Shv}_{\text{const}}(X; \mathcal{E}), Q_\zeta) \simeq L(\text{Shv}_{\text{const}}(X; \mathcal{E}), Q, \Sigma^n Q) \simeq \Sigma^n L(\text{Shv}_{\text{const}}(X; \mathcal{E}), Q) \simeq \Sigma^n (X \wedge L(\mathcal{E}, Q)).$$

This proves:

**Proposition 5.** If $\zeta$ is an oriented spherical fibration (of dimension $n$) on $X$ classified by a map $X \to \text{Pic}(S)$ which factors through $\mathbb{Z} \times \text{BPL}$, then we have homotopy equivalences (depending canonically on the choice of factorization)

$$\lim_{\tau \in T} \zeta(\tau) \wedge L^a(\mathbb{Z}) \simeq \Sigma^n (X \wedge L^a(\mathbb{Z}))$$

$$\lim_{\tau \in T} \zeta(\tau) \wedge L^s(\mathbb{Z}) \simeq \Sigma^n (X \wedge L^s(\mathbb{Z}))$$

**Remark 6.** Proposition 5 can be interpreted as saying that every orientable PL bundle is oriented with respect to the ring spectrum $L^s$. We will return to this point in the next lecture.