L-Groups of Polyhedra (Lecture 20)

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Let $\mathcal{C}$ be a stable $\infty$-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{\text{op}} \rightarrow \text{Sp}$. Let $X$ be a finite polyhedron. In the last lecture, we proved that $Q$ determines a nondegenerate quadratic functor $\text{Shv}_{\text{const}}(X; \mathcal{C})^{\text{op}} \rightarrow \text{Sp}$. Let us denote this functor by $Q_X$, to emphasize its dependence on $X$. We let $L(X; \mathcal{C}, Q)$ denote the $L$-theory space of the pair $(\text{Shv}_{\text{const}}(X; \mathcal{C}), Q_X)$.

**Example 2.** Let $X$ consists of a single point, we have $L(X; \mathcal{C}, Q) \simeq L(\mathcal{C}, Q)$.

**Remark 2.** Let $f : X \rightarrow Y$ be a map of finite polyhedra, and choose triangulations $S$ and $T$ of $X$ and $Y$ such that $f$ is linear on each simplex. Let $\mathcal{F} \in \text{Shv}_S(X; \mathcal{C})$. Then we have a canonical map

$$Q_S(\mathcal{F}) \simeq \lim_{\sigma \in S} Q(\mathcal{F}(\sigma)) \simeq \lim_{\tau \in T} Q(\mathcal{F}(f(\sigma) = \tau)) \rightarrow \lim_{\tau \in T} Q(\lim_{\sigma \in \Sigma^{-n}} \mathcal{F}(\sigma)) = Q_T(f_\ast \mathcal{F}).$$

Taking the direct limit over triangulations, we obtain a natural transformation $Q_X \rightarrow Q_Y \circ f_\ast$. This natural transformation induces a natural transformation

$$f_\ast \circ \text{VD} \rightarrow \text{VD} \circ f_\ast$$

which we showed to be an equivalence in the previous lecture.

Consequently, the pushforward functor $f_\ast$ carries quadratic objects of $\text{Shv}_{\text{const}}(X; \mathcal{C})$ to quadratic objects of $\text{Shv}_{\text{const}}(Y; \mathcal{C})$ and carries Poincare objects to Poincare objects. We obtain a map of classifying spaces

$$\text{Poinc}(\text{Shv}_{\text{const}}(X; \mathcal{C}), Q_X) \rightarrow \text{Poinc}(\text{Shv}_{\text{const}}(Y; \mathcal{C}), Q_Y).$$

The same reasoning gives a map of simplicial spaces

$$\text{Poinc}(\text{Shv}_{\text{const}}(X; \mathcal{C}), Q_X)_\ast \rightarrow \text{Poinc}(\text{Shv}_{\text{const}}(Y; \mathcal{C}), Q_Y)_\ast,$$

hence a map of $L$-theory spaces

$$L(X; \mathcal{C}, Q) \rightarrow L(Y; \mathcal{C}, Q).$$

In other words, $L(X; \mathcal{C}, Q)$ depends functorially on $X$.

We now study the functor $X \mapsto L(X; \mathcal{C}, Q)$.

**Lemma 3.** Let $n \geq 0$ be an integer, and suppose that $f, g : X \rightarrow Y$ are homotopic PL maps of finite polyhedra. Then $f$ and $g$ induce the same map $L_n(X; \mathcal{C}, Q) \rightarrow L_n(Y; \mathcal{C}, Q)$.

**Proof.** Replacing $Q$ by $\Sigma^{-n}Q$, we can reduce to the case $n = 0$. Let $(\mathcal{F}, q)$ be a Poincare object of $\text{Shv}_{\text{const}}(X; \mathcal{C})$. We wish to show that the Poincare objects $f_\ast \mathcal{F}$ and $g_\ast \mathcal{F}$ are cobordant. Choose a PL map $h : X \times [0,1] \rightarrow Y$ which is a homotopy from $f$ to $g$. Let $i_0 : X \simeq X \times \{0\} \hookrightarrow X \times [0,1]$ be the canonical map, and define $i_1$ similarly. Since the pushforward functor $h_\ast$ carries cobordisms to cobordisms, it will suffice to show that $i_{0, \ast} \mathcal{F}$ and $i_{1, \ast} \mathcal{F}$ are cobordant as Poincare objects of $\text{Shv}_{\text{const}}(X \times [0,1]; \mathcal{C})$. It now suffices to observe that a bordism between these objects is given by $p^\ast \mathcal{F}$, where $p : X \times [0,1] \rightarrow X$ denotes the projection.

From Lemma 3 we immediately deduce the following consequence:
Proposition 4. Let $f : X \to Y$ be a PL homotopy equivalence between finite polyhedra. Then $f$ induces a homotopy equivalence $L(X; \mathcal{C}, Q) \to L(Y; \mathcal{C}, Q)$.

Let $\text{Poly}$ denote the category whose objects are finite polyhedra and whose morphisms are PL maps. The construction $X \mapsto L(X; \mathcal{C}, Q)$ determines a functor from the category $\text{Poly}$ to the $\infty$-category $\mathcal{S}$ of spaces. It follows from Proposition 4 that this functor factors through $\text{Poly}[W^{-1}]$, where $\text{Poly}[W^{-1}]$ denotes the $\infty$-category obtained from $\text{Poly}$ by formally inverting all homotopy equivalences between finite polyhedra. The $\infty$-category $\text{Poly}[W^{-1}]$ is equivalent to the full subcategory $\mathcal{S}^{\text{fin}} \subseteq \mathcal{S}$ spanned by those spaces which are homotopy equivalent to a finite polyhedron (or equivalently, to a finite CW complex). We may therefore regard the functor $X \mapsto L(X; \mathcal{C}, Q)$ as defined on the $\infty$-category $\mathcal{S}^{\text{fin}}$ of finite spaces.

To continue our analysis, it will be convenient to introduce a slight variation on the above construction. Let $X$ be a finite polyhedron, and let $Y \subseteq X$ be a closed subpolyhedron. We then have a fully faithful embedding $i_* : \text{Shv}_{\text{const}}(Y; \mathcal{C}) \to \text{Shv}_{\text{const}}(X; \mathcal{C})$ which commutes with Verdier duality. It follows that the quotient $\infty$-category $\text{Shv}_{\text{const}}(X; \mathcal{C})/\text{Shv}_{\text{const}}(Y; \mathcal{C})$ inherits a nondegenerate quadratic functor. This quotient can be identified with a full subcategory of $\text{Shv}_{\text{const}}(X, Y; \mathcal{C}) \subseteq \text{Shv}_{\text{const}}(X; \mathcal{C})$: namely, the subcategory spanned by those sheaves $\mathcal{F}$ such that $i^* \mathcal{F} \simeq 0$. (Note that, for any $\mathcal{F} \in \text{Shv}_{\text{const}}(X; \mathcal{C})$, the $\infty$-category of sheaves $\mathcal{F}' \in \text{Shv}_{\text{const}}(X; \mathcal{C})$ equipped with a map $\mathcal{F}' \to \mathcal{F}$ whose cofiber is supported on $Y$ has a final object, given by the extension by zero of $\mathcal{F}|(X - Y)$.) We let $L(X, Y; \mathcal{C}, Q)$ denote the $L$-theory space of $(\text{Shv}_{\text{const}}(X, Y; \mathcal{C}), Q_X)$. We have seen that there is a fiber sequence of spaces

$$L(Y; \mathcal{C}, Q) \to L(X; \mathcal{C}, Q) \to L(X, Y; \mathcal{C}, Q).$$

More generally, for $Z \subseteq Y \subseteq Z$, we have a fiber sequence

$$L(Y, Z; \mathcal{C}, Q) \to L(X, Z; \mathcal{C}, Q) \to L(X, Y; \mathcal{C}, Q).$$

Note that the $\infty$-category $\text{Shv}_{\text{const}}(X, Y; \mathcal{C})$ can be identified with the full subcategory of $\text{Shv}_{\text{const}}(X/Y; \mathcal{C})$ spanned by those sheaves which vanish at the base point of $X/Y$. For every pointed polyhedron $Z$, let $L^{\text{red}}(Z; \mathcal{C}, Q)$ denote the relative $L$-theory space $L(Z, *; \mathcal{C}, Q)$. The construction $Z \mapsto L^{\text{red}}(Z; \mathcal{C}, Q)$ is functorial with respect to pointed PL maps between pointed finite polyhedra. Moreover, Proposition 4 implies that it carries homotopy equivalences to homotopy equivalences, and therefore extends (in an essentially unique way) to a map

$$L^{\text{red}}(\bullet; \mathcal{C}, Q) : \mathcal{S}^{\text{fin}}_\ast \to \mathcal{S},$$

where $\mathcal{S}^{\text{fin}}_\ast$ denotes the $\infty$-category of pointed finite spaces.

Proposition 5. The functor $L^{\text{red}}(\bullet; \mathcal{C}, Q) : \mathcal{S}^{\text{fin}}_\ast \to \mathcal{S}$ is excisive: that is, it carries homotopy pushout squares to homotopy pullback squares.

Proof. Consider a homotopy pushout square of finite pointed spaces

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y'.
\end{array}
$$

Without loss of generality, we may assume that each of these spaces is a finite polyhedron, each of the maps are PL, the horizontal maps are inclusions. Consider the diagram

$$
\begin{array}{ccc}
L^{\text{red}}(X; \mathcal{C}, Q) & \longrightarrow & L^{\text{red}}(X'; \mathcal{C}, Q) \\
\downarrow & & \downarrow \\
L^{\text{red}}(Y; \mathcal{C}, Q) & \longrightarrow & L^{\text{red}}(Y'; \mathcal{C}, Q)
\end{array}
\longrightarrow
\begin{array}{ccc}
L^{\text{red}}(X'/X; \mathcal{C}, Q) & \longrightarrow & L^{\text{red}}(Y'/Y; \mathcal{C}, Q) \\
\end{array}
\theta
\end{array}
$$

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Since the rows are fiber sequences, to show that the left square is a homotopy pullback, it will suffice to show that \( \theta \) is a homotopy equivalence. This is clear, since the map \( X'/X \to Y'/Y \) is a homotopy equivalence, by virtue of our assumption that \( \sigma \) is a homotopy pushout square.

It follows from Proposition 5 that we can write
\[
L^{\text{red}}(X; C, Q) \simeq \Omega^\infty(X \wedge L(C, Q))
\]
for some spectrum \( L(C, Q) \), which we will call the \textit{L-theory spectrum} of the pair \((C, Q)\). In particular, \( L(X; C, Q) \simeq L^{\text{red}}(X_+; C, Q) \) can be identified with the zeroth space of \( X_+ \wedge L(C, Q) \). Taking \( X \) to be a point, we get \( \Omega^\infty L(C, Q) = L(C, Q) \), so that the homotopy groups of the spectrum \( L(C, Q) \) are the \textit{L-groups} of the pair \((C, Q)\). More generally,
\[
L_n(X; C, Q) \simeq \pi_n(X_+ \wedge L(C, Q))
\]
is the \( n \)th homology group of \( X \) with coefficients in the spectrum \( L(C, Q) \).