L-Groups of Fields (Lecture 13)

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Our goal in this section is to carry out some calculations of $L$-groups in simple cases. We begin with the following observation:

**Proposition 1.** Let $R$ be an associative ring with involution. Then the $L$-groups of $R$ (symmetric or quadratic) are 4-periodic. That is, there are canonical isomorphisms

$$L^s_n(R) \simeq L^s_{n+4}(R) \quad L^q_n(R) \simeq L^q_{n+4}(R)$$

**Proof.** Let $\mathcal{C} = \text{LMod}^{fp}_R$. Since $\mathcal{C}$ is a stable $\infty$-category, the suspension functor is an equivalence from $\mathcal{C}$ to itself. Let $B$ be the symmetric bilinear functor given by $B(M,N) = \text{Mor}_R(M \wedge N, R)$. Then $B(\Sigma M, \Sigma N) \simeq \Sigma^{-2}B(M, N)$. Here $\Sigma^{-2}B$ is also a symmetric bilinear functor, where the symmetric group $\Sigma_2$ acts on $B$ and also on the desuspension functor $\Sigma^{-2}$ by permuting the suspension coordinates. Because the “swap” map on the sphere $S^2 = S^1 \wedge S^1$ reverses orientation, this second action is nontrivial: it acts by a sign. However, the square of this action is trivial. Consequently, we have an equivalence $B(\Sigma^2 M, \Sigma^2 N) \simeq \Sigma^{-4}B(M, N)$, compatible with the action of $\Sigma_2$ (where $\Sigma_2$ does not act on the desuspension functor $\Sigma^{-4}$).

Consequently, the double suspension map $\mathcal{C} \to \mathcal{C}$ determines equivalences

$$(\mathcal{C}, Q^s) \simeq (\mathcal{C}, Q^s_{\Sigma^2}) \quad (\mathcal{C}, Q^q) \simeq (\mathcal{C}, Q^q_{\Sigma^2}).$$

**Remark 2.** Suppose that $2 = 0$ in $R$. Then we can ignore signs. The proof of Proposition 1 then shows that the $L$-groups of $R$ are 2-periodic.

Let us now restrict our attention to the case where $R$ is a (commutative) field $k$, equipped with the trivial involution. Note that if the characteristic of $k$ is different from 2, then there is no difference between symmetric and quadratic $L$-theory. We will confine our attention to quadratic $L$-theory in what follows.

**Proposition 3.** Let $k$ be a field. Then the odd-dimensional quadratic $L$-groups $L^q_{-2m-1}(k)$ are trivial.

**Proof.** Let $(V, q)$ be a Poincare object of $(\text{LMod}^{fp}_k, \Sigma^{2m+1}Q^q)$. We wish to show that $(V, q)$ is nullcobordant. In the last lecture, we saw that we can reduce to the case where $V$ is $k$-connective. The nondegeneracy of $q$ gives an isomorphism $V \simeq \Sigma^{2m+1}\mathbb{D}(V)$. Since we are working over a field, this has concrete consequences: for every integer $i$, $\pi_i V$ is the $k$-linear dual of $\pi_{2m+1-i}(V)$. In particular, the homotopy groups $\pi_i V$ vanish for $i \notin \{m, m+1\}$. Let $W = \pi_m V$ so that $W^V \simeq \pi_{m+1} V$. Let $W[m] \in \text{LMod}^{fp}_k$ denote the module given by $W$, placed in degree $k$. Since $k$ is a field, $W$ is free as a $k$-module. We may therefore construct a map

$$\alpha : W[m] \to V$$

which induces the identity map

$$W \simeq \pi_m W[m] \to \pi_m V \simeq W.$$ 

Note that $\Sigma^{2m+1}Q^q(W[k]) \simeq (W \otimes_k W)[1]_{h\Sigma_2}$ is connected, so $q|W[m]$ is automatically nullhomotopic. Any choice of nullhomotopy exhibits $W[m]$ as a Lagrangian in $V$. □
Proposition 4. Let $k$ be a field of characteristic different from 2. Then the $L$-groups $L^q_{4m-2}(k)$ are trivial. (If $k$ has characteristic 2, then $L^q_{4m-2}(k) \simeq L^q_0(k)$ by Remark 2.)

Proof. Let $(M,q)$ be a Poincare object of $(\text{LMod}^k_p, \Sigma^{4m+2}Q^q)$. The results of the last lecture show that we can assume that $M = V[2m + 1]$ for some vector space $V$ over $k$. Let $B(V,V)$ denote the $k$-vector space of symmetric bilinear forms on $V$ (regarded as a spectrum concentrated in a single degree). Then $\Sigma^{4m+2}Q^q(M) = \Sigma^{4m+2}(\Sigma^{-4m-2}B(V,V))_{|\Sigma_2}$. Here we can ignore the distinction between invariants and coinvariants (since 2 is invertible in $k$). However, we cannot ignore the fact that $\Sigma_2$ acts nontrivially on the suspension coordinates. The upshot is that $\Sigma^{4m+2}Q^q(M)$ is the Eilenberg-MacLane spectrum corresponding to the vector space of skew-symmetric bilinear forms $b : V \times V \to k$. Since $(M,q)$ is a Poincare object, the corresponding skew-symmetric form is nondegenerate. It follows from elementary linear algebra that the dimension of $V$ must be even, and that $V$ admits a subspace $L \subseteq V$ of such that $b((L \times L)$ is trivial $\dim(V) = 2 \dim(L)$. Then $L$ is a Lagrangian in $V$, so that $(M,q)$ is nullcobordant.

Here is a slight variant on the above argument: if $V \neq 0$, then by skew-symmetry the bilinear form $b$ vanishes on the one-dimensional subspace generated by any nonzero element $v \in V$. We can therefore perform surgery to reduce the dimension of $V$. Repeat until $V \simeq 0$.)

In view of Propositions 1, 3, and 4, the calculation of the (quadratic) $L$-groups of fields reduces to the problem of understanding the group $L^d_0(k)$. This is an interesting classical invariant.

Definition 5. Let $k$ be a field. A quadratic space over $k$ is a pair $(V,q)$, where $V$ is a finite-dimensional vector space over $k$ and $q : V \to k$ is a quadratic form. That is, $q$ satisfies

\[ q(ax) = a^2q(x) \quad q(x + y) = q(x) + q(y) + b(x,y) \]

for some bilinear form $b : V \times V \to k$. We say that $q$ is nondegenerate if $b$ is nondegenerate.

Example 6. Let $k$ be any field. There is a quadratic space $H = (k^2, q)$ over $k$, where $q$ is given by the formula $q(a,b) = ab$. We refer to $H$ as the hyperbolic plane.

There is an evident direct sum operation on quadratic spaces: given a pair of quadratic spaces $(V,q)$ and $(V',q')$, we define $(V,q) \oplus (V',q')$ to be $(V \oplus V', q \oplus q')$, where $q \oplus q' : V \oplus V' \to k$ is given by the formula

\[ (q \oplus q')(v,v') = q(v) + q'(v'). \]

Remark 7. Let $(V,q)$ be a nondegenerate quadratic space over a field $k$. Suppose we are given a nonzero element $x \in V$ such that $q(x) = 0$. Since the associated bilinear form $b$ is nondegenerate, we can choose $y \in V$ with $b(x,y) = 1$. Note that $b(x,x) = q(2x) = 2q(x) = 0$. It follows that $q(y + ax) = q(y) + ab(y,x) + q(ax) = q(y) + a$. In particular, $q(y - q(y)x) = 0$. Replacing $y$ by $y - q(y)x$, we can reduce to the case where $q(y) = 0$. Then if $V_0$ denotes the subspace of $V$ generated by $x$ and $y$, then we have an isomorphism $(V_0, q|V_0) \simeq H$. In particular, $q$ is nondegenerate on $V_0$ and we therefore have a decomposition $(V,q) \simeq H \oplus (V_1,q|V_1)$, where $V_1$ is the orthogonal complement of $V_0$.

More generally, if we are given a subspace $W \subseteq V$ of dimension $a$ such that $q|W = 0$, we can apply this argument repeatedly to obtain a decomposition $(V,q) \simeq H \oplus (V', q')$.

Definition 8. Let $k$ be a field. We say that two nondegenerate quadratic spaces $(V,q)$ and $(V',q')$ are stably equivalent if $(V,q) \oplus H^{\oplus a}$ is isomorphic to $(V',q') \oplus H^{\oplus b}$ for some integers $a$ and $b$. The collection of stable equivalences classes of nondegenerate quadratic spaces over $k$ is called the Witt group of $k$. We will denote it by $W(k)$ (not to be confused with the ring of Witt vectors over $k$).

The set $W(k)$ evidently has the structure of a commutative monoid under direct sum. In fact, this monoid structure is a group: for any nondegenerate quadratic space $(V,q)$ where $V$ has dimension $d$, the sum $(V,q) \oplus (V,-q)$ has an isotopic subspace of dimension $d$ (the image of $V$ under the diagonal map $V \to V \oplus V$) and is therefore isomorphic to $H^{\oplus d}$ by Remark 7.
Remark 9. Let \((V, q)\) be any nondegenerate quadratic space over \(k\). Using Remark 7 repeatedly, we deduce that \((V, q)\) is isomorphic to a direct sum \((V', q') \oplus H^{\otimes d}\) for some integer \(d\), where \((V', q')\) is anisotropic: that is, \(q'\) does not vanish on any nonzero element of \(V'\). Consequently, every class in the Witt group \(W(k)\) can be represented by an anisotropic quadratic space \((V, q)\). In fact, this representative is unique up to isomorphism. This is a consequence of the Witt cancellation theorem, which asserts that if we have an isomorphism of nondegenerate quadratic spaces

\[(V, q) \oplus (V'', q'') \simeq (V', q') \oplus (V'', q''),\]

then \((V, q)\) and \((V', q')\) must already be isomorphic.

Let \((V, q)\) be a nondegenerate quadratic space over a field \(k\). Viewing \(V\) as a chain complex over \(k\) concentrated in degree zero, we can think of \((V, q)\) as a Poincare object of \((\text{LMod}^p_k, Q^0)\). This construction determines a map \(W(k) \to L_0^k(k)\).

Proposition 10. Let \(k\) be a field. Then the map \(\phi : W(k) \to L_0^k(k)\) is an isomorphism of abelian groups.

Proof. We have already seen that \(\phi\) is surjective (using surgery below the middle dimension). Let us show that \(\phi\) is injective. Let \((V, q)\) be a quadratic space over \(k\), and suppose that there exists a Lagrangian in \(V\) (as a Poincare object of \((\text{LMod}^p_k, Q^0)\)). Denoting this Lagrangian by \(L\), we have a fiber sequence of spectra

\[L \xrightarrow{\alpha} V \to \text{cofib}(\alpha)\]

which is self-dual (with the duality on \(V\) determined by \(q\)). In particular, we have a self-dual short exact sequence of vector spaces

\[0 \to (\text{Im} \pi_0 L \to V) \to V \to (\text{Im} V \to \pi_0 \text{cofib}(\alpha)) \to 0.\]

The self-duality implies that the dimensions of the outer two vector spaces are the same, so that the dimension of \(V\) is twice as large as the dimension of \(W = \text{Im}(\pi_0 L \to V)\). The map \(W \to V\) factors through \(L\), so \(qW = 0\). Using Remark 7, we deduce that \(V\) is isomorphic to a direct sum of hyperbolic planes so that \((V, q)\) is equivalent to zero in the Witt group \(W(k)\).

\(\square\)

Example 11. Let \(k = F_2\) be the finite field with two elements. Let \((V, q)\) be a nondegenerate quadratic space over \(k\). Then the dimension of \(V\) must be even (since the symmetric bilinear form \(b\) is also a nondegenerate skew-symmetric bilinear form). Suppose that \(V\) is anisotropic: then \(q(v) = 1\) for every nonzero element \(v \in V\). It follows that if \(v, w \in W\) are distinct and nonzero, then \(b(v, w) = q(v + w) - q(v) - q(w) = 1\). If \(u, v, w \in W\) are linearly independent, we get

\[1 = b(u, v + w) = b(u, v) + b(u, w) = 0.\]

Thus any nontrivial anisotropic quadratic space must be of dimension 2. There is such a space \((V, q)\): take \(V = F_2 \oplus F_2\), and \(q\) to be given by the formula

\[q(a, b) = a^2 + ab + b^2.\]

It follows from the Witt cancellation theorem that \((V, q)\) determines a nontrivial element of \(W(k)\) (this can also be deduced by evaluating some of the invariants introduced below). We therefore have an isomorphism \(W(k) \simeq \mathbb{Z}/2\mathbb{Z}\).

To any nondegenerate quadratic space \((V, q)\) over \(k = F_2\), we can associate an invariant in the group \(W(k = \mathbb{Z}/2\mathbb{Z})\). This is called the Arf invariant of \((V, q)\). It can be described concretely as follows: the Arf invariant of \(q\) is 0 if \(q\) takes the value 0 more often than 1 (that is, if the set \(q^{-1}(0) \subseteq V\) is larger than the set \(q^{-1}(1) \subseteq V\), and takes the value 1 otherwise. A more conceptual description of this invariant is given below.
Example 12. Let $k$ be an algebraically closed field. Any two nondegenerate quadratic spaces $(V, q)$ over $k$ of the same dimension are isomorphic. It follows that $W(k)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if the characteristic of $k$ is different from 2, and is trivial if the characteristic of $k$ is equal to 2 (since any nondegenerate quadratic space must be even dimension in the latter case).

Example 13. Let $k$ be the field of real numbers (or any real-closed field). Then Sylvester’s invariance of signature theorem gives an isomorphism of abelian groups $W(k) \simeq \mathbb{Z}$, which carries a nondegenerate quadratic space $(V, q)$ to the signature $(\sigma(q))$ of even dimension over $k$.

Remark 14. Let $k$ be a field of characteristic $\neq 2$. For every nonzero element $a \in k$, we have a nondegenerate quadratic form $q : k \to k$ given by $x \mapsto ax^2$. We denote the image of this element in $W(k)$ by $(a)$. These elements generate $W(k)$ under addition, because any nondegenerate quadratic space $(V, q)$ has an orthogonal basis. It is possible to explicitly write down a set of relations between these generators, and thereby obtain a presentation for $W(k)$.

We now describe some invariants that can help get a handle on the structure of a Witt ring $W(k)$. We first note that since the hyperbolic plane $H$ has dimension 2, every element of $W(k)$ has a well-defined dimension modulo 2. This yields a group homomorphism

$$d : W(k) \to \mathbb{Z}/2\mathbb{Z}.$$ 

The map $d$ is surjective when $k$ has characteristic different from 2, and is the zero map when $k$ has characteristic 2. Let $I$ denote the kernel of $d$.

Suppose we are given an element of $I \subseteq W(k)$. We can represent this element by a nondegenerate quadratic space $(V, q)$ of even dimension over $k$. We define the Clifford algebra $Cl(V, q)$ to be the quotient of the free associative $k$-algebra on $V$ by the relations

$$x^2 = q(x)$$

for $x \in V$. This Clifford algebra has a canonical $\mathbb{Z}/2\mathbb{Z}$-grading

$$Cl(V, q) \simeq Cl_0(V, q) \oplus Cl_1(V, q),$$

where we take the elements of $V$ to have degree 1. If $V \neq 0$, then one can show that the center of $Cl_0(V, q)$ is a rank 2 étale extension of $k$: that is, it is either isomorphic to $k \times k$ or to a separable quadratic extension field $k'$ of $k$. This extension of $k$ determines a map $\text{Gal}(\overline{k}/k) \to \mathbb{Z}/2\mathbb{Z}$, (which is the zero map if and only if the center is isomorphic to $k \times k$). The formation of this invariant determines a group homomorphism

$$\psi : I \to H^1(\text{Gal}(\overline{k}/k); \mathbb{Z}/2\mathbb{Z}).$$

The image of a quadratic space $(V, q)$ under this map is called the *discriminant* of $(V, q)$. When the characteristic of $k$ is different from 2, we can identify the discriminant with an element in $k^\times / (k^\times)^2$ (using Kummer theory). When $k$ has characteristic 2, we can identify the discriminant with an element of the cokernel of the map

$$k \xrightarrow{x \mapsto x^2 - 1} k$$

(using Artin-Schreier theory). When $k = \mathbb{F}_2$, we recover the Arf invariant discussed above.

Let $J \subseteq W(k)$ denote the kernel of the map $\psi$ defined above. Elements of $J$ can be represented by quadratic spaces $(V, q)$ of even dimension such that the center of $Cl_0(V, q)$ splits as a product $k \times k$. It follows that $Cl_0(V, q)$ itself splits as a product of two factors. One can show that each of these factors is a central simple algebra over $k$, and determines a 2-torsion element in the Brauer group of $k$. Let us assume that $k$ has characteristic different from 2, so we can identify $\mathbb{Z}/2\mathbb{Z}$ with the subset $\{1, -1\} \subseteq k^\times$. Extracting these Brauer invariants gives a homomorphism

$$J \to H^2(\text{Gal}(\overline{k}/k); \mathbb{Z}/2\mathbb{Z}).$$
This pattern continues. If the characteristic of $k$ is different from 2, then the Witt group $W(k)$ actually has the structure of a ring (given symmetric bilinear forms on vector spaces $V$ and $W$, we obtain a symmetric bilinear form on $V \otimes W$). The map $d : W(k) \to \mathbb{Z}/2\mathbb{Z}$ is a ring homomorphism, so that $I \subseteq W(k)$ is an ideal. The following is a deep result of Voevodsky:

**Theorem 15** (The Milnor Conjecture). *If $k$ is a field of characteristic different from 2, there are canonical isomorphisms*

$$
I^m/I^{m+1} \cong H^m(\text{Gal}(\overline{k}/k); \mathbb{Z}/2\mathbb{Z})
$$