Before getting to the main topic of this lecture, let us pick up a few loose ends from the previous lecture. Recall that for every $A_\infty$-ring $R$ with involution, we defined stable $\infty$-categories $\text{LMod}^{\text{perf}}_R$ and $\text{LMod}^\text{fp}_R$ which are equipped with quadratic functors $Q$ and $Q^\text{symm}$. We ask the following question: how general are these examples, among all pairs $\left(\mathcal{C}, Q\right)$ where $\mathcal{C}$ is a stable $\infty$-category and $Q$ a nondegenerate quadratic functor on $\mathcal{C}$?

- Let $\mathcal{C}$ be any stable $\infty$-category containing an object $X$. Then the spectrum $\text{Mor}_\mathcal{C}(X, X)$ is an $A_\infty$-ring spectrum $R$. Moreover, the construction $M \mapsto X \wedge_R M$ determines a fully faithful embedding $\text{LMod}^\text{fp}_R \to \mathcal{C}$, carrying $R$ to $X$. The essential image of this functor is the smallest stable subcategory of $\mathcal{C}$ containing $X$. If $\mathcal{C}$ is idempotent complete, then this functor extends to a fully faithful embedding $\text{LMod}^{\text{perf}}_R \to \mathcal{C}$.

- Suppose now that $\mathcal{C}$ is equipped with a symmetric bilinear functor $B$, and let $Q: \mathcal{C}^{\text{op}} \to \text{Sp}$ be given by the formula $Q(X) = B(X, X)^{h\Sigma_2}$. Let us assume that $B$ is nondegenerate, and denote the associated duality functor by $D$. The functor $D$ is a contravariant equivalence from $\mathcal{C}$ to itself. Consequently, for any object $X \in \mathcal{C}$ we have a canonical equivalence of $A_\infty$-rings $\text{Mor}_\mathcal{C}(X, X) \simeq \text{Mor}_\mathcal{C}(D(X), D(X))^{op}$. If $(X, q)$ is a Poincare object of $\mathcal{C}$, then this equivalence determines an involution $\sigma$ on the $A_\infty$-ring $R = \text{Mor}_\mathcal{C}(X, X)$.

We can summarize the above discussion as follows: a pair $\left(\mathcal{C}, Q\right)$ is of the form $\left(\text{LMod}^\text{fp}_R, Q\right)$ if and only if $Q(X) \simeq B(X, X)^{h\Sigma_2}$ for $X \in \mathcal{C}$, and there exists a Poincare object $(M, q)$ in $\mathcal{C}$ such that $M$ generates $\mathcal{C}$ as a stable $\infty$-category.

**Example 1.** Let $R$ be an associative ring, and let $D^\text{fp}(R)$ be the $\infty$-category of bounded chain complexes of finitely generated free modules. We can regard $R$ as an object of $D^\text{fp}(R)$. Let $A = \text{Mor}(R, R)$. The homotopy groups of $A$ are then given by the formula

$$
\pi_i A = \text{Ext}^{-i}_R(R, R) = \begin{cases} 
R & \text{if } i = 0 \\
0 & \text{if } i \neq 0.
\end{cases}
$$

In other words, we can identify $A$ with the discrete $A_\infty$-ring corresponding to $R$. Following the above outline, we obtain a fully faithful embedding $\text{LMod}^\text{fp}_R \to D^\text{fp}(R)$. Since $D^\text{fp}(R)$ is generated by $R$ (under taking fibers and cofibers), we see that this fully faithful embedding is an equivalence.

Our goal in the next few lectures is to obtain a concrete description of the quadratic $L$-theory for a ring $R$ with involution (and, more generally, for an $A_\infty$-ring with involution). The obstacle we have to overcome is this: by definition, elements of $L_0(R)$ are represented by arbitrary finite complexes of free $R$-modules $P_*$, equipped with a quadratic form represented by a cycle $q$ in $\text{Hom}(P_* \otimes P_*, R)^{h\Sigma_2}$. We saw in the last lecture that there is a concrete description of what it means to give a cycle, at least in the special case where $R$ is a commutative ring with trivial involution and $P_*$ is concentrated in a single degree. In this lecture, we describe a mechanism which can be used to show that an arbitrary Poincare object $(P_*, q)$ is cobordant
In other words, the composite map is canonically nullhomotopic. We therefore obtain a triangle concentrated in a single degree. The cobordism itself will be constructed using the method of surgery. Let us begin in a general setting. Let \( \mathcal{C} \) be a stable \( \infty \)-category, \( Q : \mathcal{C}^{\text{op}} \to \text{Sp} \) a nondegenerate quadratic functor with polarization \( B \), and \( \mathbb{D} \) the associated duality functor. Suppose we are given a fiber sequence

\[
X' \xrightarrow{\alpha} X \to X/X'
\]

in \( \mathcal{C} \). Let \( q \in \Omega^\infty Q(X) \), and suppose that we are given a nullhomotopy of \( q|X' \in \Omega^\infty Q(X') \). We have seen that this is generally not enough information to allow us to descend \( q \) to a point of \( \Omega^\infty Q(X/X') \), because \( q \) may have nontrivial image \( b \in \Omega^\infty B(X', X) \). Note however that \( q \) does have trivial image in \( \Omega^\infty B(X', X') \). In other words, the composite map

\[
X' \xrightarrow{\alpha} X \to \mathbb{D}(X) \xrightarrow{\mathbb{D}(\alpha)} \mathbb{D}(X')
\]

is canonically nullhomotopic. We therefore obtain a triangle

\[
X' \xrightarrow{\alpha} X \xrightarrow{\beta} \mathbb{D}(X)
\]

in the stable \( \infty \)-category \( \mathcal{C} \). In general, this triangle is not a fiber sequence. Its failure to be a fiber sequence can be measured by taking homology: that is, by extracting the object

\[
\text{cofib}(X \to \text{fib}(\beta)) \simeq \text{fib}(\text{cofib}(\alpha) \to \mathbb{D}(X))
\]

of \( \mathcal{C} \) (which vanishes if and only if the sequence above is a fiber sequence). Let us denote this homology object by \( X_\alpha \). This is abusive: it depends not only on \( \alpha \), but on a choice of nullhomotopy of \( q|X' \).

Let us write \( X_\alpha = \text{fib}(\beta)/X \). We have seen that there is a fiber sequence

\[
Q(X_\alpha) \to Q(\text{fib}(\beta)) \to Q(X') \times_{B(X', X')} B(\text{fib}(\beta), X').
\]

The point \( q \) determines a point of \( \Omega^\infty B(X, X') \), classifying the map \( \beta : X \to \mathbb{D}X' \). By construction, this map is canonically nullhomotopy after composition with the map \( \text{fib}(\beta) \to X \). Consequently, the restriction \( q|\text{fib}(\beta) \) has trivial image in \( \Omega^\infty (Q(X') \times_{B(X', X')} B(\text{fib}(\beta), X')) \), and therefore lifts to a point \( q_\alpha \in \Omega^\infty Q(X_\alpha) \).

We may therefore view \( (X_\alpha, q_\alpha) \) as another quadratic object of \( (\mathcal{C}, Q) \). We say that \( (X_\alpha, q_\alpha) \) is obtained from \( (X, q) \) via surgery on \( \alpha \).

The point \( q_\alpha \) determines a map from \( X_\alpha \) to its dual. This map can be described more explicitly as follows. Note that if we are given a triangle

\[
Y' \to Y \to Y''
\]

in \( \mathcal{C} \), then we can dualize to obtain a new triangle

\[
\mathbb{D}(Y'') \to \mathbb{D}(Y) \to \mathbb{D}(Y')
\]

in \( \mathcal{C} \). The process of extracting homology is self-dual (rather, the two descriptions of homology given above are dual to one another). The map \( X_\alpha \to \mathbb{D}X_\alpha \) is given by the map on homology induced by a map of triangles

\[
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow & & \downarrow \\
\mathbb{D}^2(X') & \rightarrow & \mathbb{D}(X) \\
\downarrow & & \downarrow \\
\mathbb{D}(X') & \rightarrow & \mathbb{D}(X').
\end{array}
\]

Here the outer maps vertical maps are isomorphisms and the middle map is induced by \( q \). Consequently, the cofiber of the map \( X_\alpha \to \mathbb{D}(X_\alpha) \) is given by the homology of the triangle

\[
0 \to \text{cofib}(X \to \mathbb{D}(X)) \to 0,
\]

which is the same as the cofiber of the map \( X \to \mathbb{D}(X) \). In particular, one cofiber vanishes if and only if the other does. We have proven:
Proposition 2. Let \((X,q)\) be a Poincare object of \(\mathcal{C}\). Suppose we are given a map \(\alpha : X' \to X\) and a nullhomotopy of \(q|X'\), and let \((X_\alpha,q_\alpha)\) be obtained by surgery along \(\alpha\). Then \((X_\alpha,q_\alpha)\) is also a Poincare object of \(\mathcal{C}\).

In fact, we can say more. By construction, \(q_\alpha\) and \(q\) have the same restriction to \(L = \text{fib}(X \to \mathbb{D}(X'))\). The identification of these restrictions determines a map
\[
X' \simeq \text{fib}(L \to X_\alpha) \to \text{cofib}(L \to X) = \mathbb{D}(\mathbb{D}(X')) \simeq X'.
\]

Unwinding the definitions, one shows that this map is the identity up to a sign. Consequently, the Poincare object \((X,q)\) and \((X_\alpha,q_\alpha)\) are cobordant, and determine the same element of the abelian group \(L_0(\mathcal{C},Q)\).

Remark 3. With some additional effort, one can show that all cobordisms arise via this construction. That is, every Poincare object cobordant to \((X,q)\) has the form \((X_\alpha,q_\alpha)\), for some map \(\alpha : X' \to X\) and some nullhomotopy of \(q|X'\).