April 10, 2014

Let $k$ be an algebraically closed field, let $X$ be an algebraic curve over $k$, and let $G$ be a smooth affine group scheme over $X$.

In the last lecture, we introduced a family of prestacks \{Ran\textsuperscript{†}_G(X)_S\}_{S \in \text{Fin}^*} equipped with maps $\phi_S : \text{Ran}_G(X)_S \to \text{Ran}(X)$, and a family of !-sheaves $\mathcal{B}_S$ given informally by the formulae

$$\mathcal{B}_S = [\text{Ran}_G(X)_S]_{\text{Ran}(X)} = \phi_S^* \phi_S^* \omega_{\text{Ran}(X)}.$$

Our goal for the next several lectures is to prove the following:

**Proposition 1.** Assume that the generic fiber of $G$ is semisimple and simply connected. Then the canonical maps \{Ran\textsuperscript{†}_G(X)_S \to \text{Ran}_G(X)_S\}_{S \in \text{Fin}^*} induce an equivalence

$$\mathcal{B} \to \lim_{S \in \text{Fin}^*} \mathcal{B}_S$$

in the $\infty$-category $\text{Shv}^! (\text{Ran}(X))$.

By its nature, Proposition 1 is “local” on $\text{Ran}(X)$. To prove it, it will suffice to show that for every nonempty finite set $T$, the underlying map

$$\theta_T : [\text{Ran}_G(X)_T \times_X Y]_T \to \lim_{S \in \text{Fin}^*} [\text{Ran}_G(X)_S \times_X Y]_T$$

is an equivalence in $\text{Shv}_T(X_T)$, where $\text{Ran}_G(X)_T$ denotes the fiber product $\text{Ran}_G(X) \times_{\text{Ran}(X)} X_T$, and $\text{Ran}_G(X)_S$ is defined similarly. In fact, we will prove the following stronger assertion:

**Proposition 2.** Let $T$ be a nonempty finite set, fixed throughout this lecture. Let $Y$ be a quasi-projective $k$-scheme equipped with a map $Y \to X_T$. Then the canonical map

$$\theta_Y : [\text{Ran}_G(X)_T \times_X Y]_Y \to \lim_{S \in \text{Fin}^*} [\text{Ran}_G(X)_S \times_X Y]_Y$$

is an equivalence in $\text{Shv}_T(Y)$.

The virtue the formulation given in Proposition 2 is that it will allow us to apply a devissage to the scheme $Y$. Suppose we are given a pullback diagram of $k$-schemes

$$
\begin{array}{ccc}
U' & \longrightarrow & U \\
g' \downarrow & & \downarrow g \\
Y' & \longrightarrow & Y,
\end{array}
$$

where the horizontal maps are proper. We have seen that this diagram induces a map $[U']_{Y'} = g'_* g'^* \omega_{Y'} \to f'^! g_* g^* \omega_Y = f'^! [U]_Y$. If the vertical maps are smooth, then the smooth base change theorem implies that this
map is invertible: that is, we can identify \([U']_Y'\) with \(f'[U]_Y'\). One can show that this holds more generally for commutative diagrams of prestacks

\[
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{g'} & \mathcal{C} \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{f} & Y,
\end{array}
\]

provided satisfying one of the following conditions:

(a) The map \(g\) exhibits \(\mathcal{C}\) as an Artin stack which is smooth over \(Y\) (this condition is satisfied by the morphisms \(\text{Ran}^G(X)^T \times_{X^T} Y \to Y\)).

(b) The prestack \(\mathcal{C}\) admits an open immersion into a product \(\mathcal{C}_0 \times \text{Spec}kY\). (This condition is satisfied by the morphisms \(\text{Ran}^G(X)_{\overline{\mathcal{C}}} \times_{X^T} Y \to Y\).

It follows that for any proper map \(f : Y' \to Y\) of \(X^T\)-schemes, we can identify \(\theta_Y\) with the image of \(\theta_Y\) under the functor \(f^! : \text{Shv}_I(Y) \to \text{Shv}_I(Y')\).

**Remark 3.** Suppose that \(i : Y' \to Y\) is a closed immersion, with complementary open immersion \(j : U \to Y\). For any object \(\mathcal{F} \in \text{Shv}_{I}(Y)\), we have a canonical fiber sequence

\[
i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F}.
\]

In particular, \(\mathcal{F} \simeq 0\) if and only if both \(i^!\mathcal{F}\) and \(j^*\mathcal{F}\) vanish. It follows that \(\theta_Y\) is an equivalence if and only if \(i^!(\theta_Y) \simeq \theta_{Y'}\) and \(j^*(\theta_Y) \simeq \theta_U\) are equivalences.

The proof of Proposition 2 will proceed by Noetherian induction on \(Y\). That is, to prove that \(\theta_Y\) is an equivalence, we may assume without loss of generality that \(\theta_{Y'}\) is an equivalence for every closed subscheme \(Y' \subseteq Y\). If \(Y\) is non-reduced, we can complete the proof by taking \(Y' = Y_{\text{red}}\). Let us assume that \(Y\) is nonempty (otherwise, there is nothing to prove). By virtue of Remark 3, it will suffice to prove Proposition 2 after replacing \(Y\) by an arbitrary nonempty open subset of \(Y\). We may therefore assume without loss of generality that \(Y = \text{Spec} R\) is smooth and affine. In this case, the map \(Y \to X^T\) corresponds to a map \(\nu : T \to X(R)\).

**Remark 4.** In class, we will eventually specialize to the case where the group scheme \(G\) is split reductive (in which case the proof becomes dramatically simpler). If this condition were not satisfied, it would be convenient at this point to assume in addition that the map \(Y \to X^T\) is “transverse” to \(G\): that is, that each of the maps \(\nu(t) : \text{Spec} R \to X\) is either constant or has image disjoint from the locus where \(G\) is not reductive.

Let us say that an object \(\mathcal{F} \in \text{Shv}_{I}(Y)\) is \(\ell\)-adically complete if limit of the tower

\[
\cdots \to \mathcal{F} \xrightarrow{\ell} \mathcal{F} \xrightarrow{\ell} \mathcal{F}
\]

vanishes. Equivalently, \(\mathcal{F}\) is \(\ell\)-adically complete if it can be recovered as the limit of the tower

\[
\cdots \to (\mathbb{Z}/\ell^d\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathcal{F} \to (\mathbb{Z}/\ell^2\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathcal{F} \to (\mathbb{Z}/\ell\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathcal{F}.
\]

Any constructible sheaf is \(\ell\)-adically complete, and the collection of \(\ell\)-adically complete objects of \(\text{Shv}_{I}(Y)\) is closed under limits. It follows that for every map of prestacks \(\mathcal{C} \to Y\), the sheaf \([\mathcal{C}]_Y \in \text{Shv}_{I}(Y)\) is \(\ell\)-adically complete. In particular, \(\theta_Y\) is a morphism between \(\ell\)-adically complete objects of \(\text{Shv}_{I}(Y)\). Consequently, to prove that \(\theta_Y\) is an equivalence, it will suffice to show that \(\theta_Y\) induces an equivalence after tensoring with \(\mathbb{Z}/\ell^d\mathbb{Z}\), for every integer \(d \geq 0\). In other words, it will suffice to prove the analogue of Proposition 2 after
replacing $\mathbb{Z}_\ell$ by $\mathbb{Z}/\ell^d\mathbb{Z}$. Note that we can detect equivalences in $\text{Shv}(Y; \mathbb{Z}/\ell^d\mathbb{Z})$ by passing to global sections over étale $Y$-schemes $V$. Replacing $Y$ by $V,$ we are reduced to proving that the canonical map 

$$[\text{Ran}^G(X)^T \times_X Y]_Y(Y) \to \lim_{S}[\text{Ran}^G_{\mathbb{O}}(X)^T_S \times_X Y]_Y(Y)$$

is a quasi-isomorphism. Since $Y$ is smooth, the dualizing complex $\omega_Y$ agrees with the constant sheaf on $Y$ up to a shift, so that we identify $[\mathcal{C}]_Y(Y)$ with a shift of $C^*(\mathcal{C}; \mathbb{Z}/\ell^d\mathbb{Z})$ for any prestack $\mathcal{C}$ over $Y$. We are therefore reduced to proving that the canonical map

$$C^*(\text{Ran}^G(X)^T \times_X Y; \mathbb{Z}/\ell^d\mathbb{Z}) \to \lim_{S} C^*(\text{Ran}^G_{\mathbb{O}}(X)^T_S \times_X Y; \mathbb{Z}/\ell^d\mathbb{Z})$$

is an equivalence in $\text{Mod}_{\mathbb{Z}/\ell^d\mathbb{Z}}$. In fact, we will prove a stronger assertion at the level of homology. For simplicity, let us henceforth assume that the group scheme $G$ is constant.

**Proposition 5.** Suppose we are given a map $Y = \text{Spec} R \to X^T,$ corresponding to a map $\nu : T \to X(R)$ which is in general position. Then the canonical map

$$\lim_{S} C_*(\text{Ran}^G_{\mathbb{O}}(X)^T_S \times_X Y; \mathbb{Z}_\ell) \to C_*(\text{Ran}^G(X)^T \times_X Y; \mathbb{Z}_\ell)$$

is an equivalence in $\text{Mod}_{\mathbb{Z}_\ell}.$

**Remark 6.** Proposition 5 can be generalized to the case of a non-constant group scheme $G$, but the notion of "general position" needs to be slightly modified.

Let us now outline our strategy for proving Proposition 5. First, $\text{Ran}^G_{\mathbb{O}}(X)^T$ denote the prestack obtained by applying the Grothendieck construction to the functor $S \mapsto \text{Ran}^G_{\mathbb{O}}(X)^T_S$. More precisely, $\text{Ran}^G_{\mathbb{O}}(X)^T$ denotes the category whose objects are tuples $(A, S, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma)$ where $A$ is a finitely generated $k$-algebra, $S$ is a nonempty finite set, $K_-$ and $K_+$ are subsets of $S$ with $K_- \subseteq K_+$, $\mu : S \to X(A)$ and $\nu : T \to X(A)$ are maps such that $|\mu(K_-)|$ and $|\nu(T)|$ do not intersect, $\mathcal{P}$ is a $G$-bundle on $X_A - |\mu(K_-)|$ which can be extended to a $G$-bundle on $X_A$, and $\gamma$ is a trivialization of $\mathcal{P}$ over $X_A - |\mu(S)|$. Then we can identify the direct limit $\lim_{S} C_*(\text{Ran}^G_{\mathbb{O}}(X)^T_S \times_X Y; \mathbb{Z}_\ell)$ with $C_*(\text{Ran}^G_{\mathbb{O}}(X)^T \times_X Y; \mathbb{Z}_\ell)$. It will therefore suffice to show that the forgetful functor

$$\text{Ran}^G_{\mathbb{O}}(X)^T \times_X Y \to \text{Ran}^G(X)^T \times_X Y$$

induces an isomorphism on homology. To prove this, we will need an auxiliary constructions:

**Definition 7.** We define a category $\text{Ran}^G_{\text{germ}}(X)^T$ as follows:

(a) The objects of $\text{Ran}^G_{\text{germ}}(X)^T$ are triples $(A, \nu, \mathcal{P})$ where $A$ is a finitely generated $k$-algebra, $\nu : T \to X(A)$ is a map, and $\mathcal{P}$ is a $G$-bundle on $X_A$.

(b) A morphism from $(A, \nu, \mathcal{P})$ to $(A', \nu', \mathcal{P}')$ is a $k$-algebra homomorphism $A \to A'$ such that $\nu'$ coincides with the composite map $T \overset{\nu}{\to} X(A) \to X(A')$, together with a germ of $G$-bundle isomorphisms of $X_A' \times_X A' \mathcal{P}$ with $\mathcal{P}'$ around the divisor $|\nu'| \subseteq X_A'$ (that is, we require an isomorphism which is defined on some open subset of $X_A'$ which contains $|\nu'|$).

We have evident forgetful functors

$$\text{Ran}^G_{\mathbb{O}}(X)^T \to \text{Ran}^G_{\text{germ}}(X)^T \to \text{Ran}^G(X)^T.$$

To prove Proposition 5, it will suffice to show that for every map $Y = \text{Spec} R \to X^T$, the maps

$$\text{Ran}^G_{\mathbb{O}}(X)^T \times_X Y \xrightarrow{\ell^d} \text{Ran}^G_{\text{germ}}(X)^T \times_X Y \xrightarrow{\ell^d} \text{Ran}^G(X)^T \times_X Y$$

induce isomorphisms on $\mathbb{Z}_\ell$-homology. We will take this up in the next lecture.