Let \( k \) be an algebraically closed field, let \( X \) be an algebraic curve over \( k \), and let \( G \) be a smooth affine group scheme over \( X \). Let us assume for simplicity that \( G \) is everywhere reductive. Associated to \( G \), we have prestack morphisms

\[
\text{Ran}_G(X) \overset{\phi}{\to} \text{Ran}(X)
\]

\[
\text{Ran}^G(X) \overset{\psi}{\to} \text{Ran}(X).
\]

Recall that the objects of \( \text{Ran}_G(X) \) are tuples \((R,S,\mu,\mathcal{P},\gamma)\) where \( R \) is a finitely generated \( k \)-algebra, \( S \) is a nonempty finite set, \( \mu : S \to X(R) \) is a map, \( \mathcal{P} \) is a \( G \)-bundle on \( X_R \), and \( \gamma \) is a trivialization of \( \mathcal{P} \) on \( X - |\mu| \). The objects of \( \text{Ran}^G(X) \) are tuples \((R,T,\nu,\mathcal{P})\) where \( R \) is a finitely generated \( k \)-algebra, \( T \) is a nonempty finite set, \( \nu : T \to X(R) \) is a map, and \( \mathcal{P} \) is a \( G \)-bundle on \( X - |\nu| \). These two maps have different variance properties. Given a nonempty finite set \( S \), the projection map

\[
\text{Ran}_G(X)_S = \text{Ran}_G(X) \times_{\text{Ran}(X)} X^S \to X^S
\]

is Ind-proper (since \( G \) is everywhere reductive); for a nonempty finite set \( T \), the map

\[
\text{Ran}^G(X)^T = \text{Ran}^G(X) \times_{\text{Ran}(X)} X^T \to X^T
\]

is instead a smooth morphism of algebraic stacks. Given a surjection of nonempty finite sets \( S \to S' \), we have a natural map

\[
\text{Ran}_G(X)_{S'} \to X^{S'} \times_{X^S} \text{Ran}_G(X)_S,
\]

while a surjection \( T \to T' \) instead induces a map

\[
\text{Ran}^G(X)^{T'} \leftarrow X^{T'} \times_{X^T} \text{Ran}^G(X)^T.
\]

As a consequence of these differences, the maps \( \phi \) and \( \psi \) can be used to produce two different types of sheaves on \( \text{Ran}(X) \). We have previously defined the !-sheaf \( \mathcal{B} = [\text{Ran}^G(X)]_{\text{Ran}(X)} \), which we can think of informally as given by \( \psi_* \psi^* \omega_{\text{Ran}(X)} \). Similarly, one can construct a *-sheaf \( \mathcal{A} = \phi_* \phi^* \mathcal{Z}_{\text{Ran}(X)} \). Our goal in this lecture is to describe how these constructions are related. As we have hinted earlier, these objects are related by a covariant form of Verdier duality, at least after an appropriate normalization.

We begin with an informal discussion. Consider the prestack \( \text{Ran}(X) \times_{\text{Spec} \ k} \text{Ran}(X) \). Let us informally identify the points of \( \text{Ran}(X) \times_{\text{Spec} \ k} \text{Ran}(X) \) with pairs \((S,T)\), where \( S \) and \( T \) are nonempty finite subsets of \( X \). Given such a pair, any \( G \)-bundle \( \mathcal{P} \) on \( X \) can be restricted to a \( G \)-bundle on \( T \). This construction determines a commutative diagram

\[
\text{Ran}_G(X) \times_{\text{Spec} \ k} \text{Ran}(X) \longrightarrow \text{Ran}(X) \times_{\text{Spec} \ k} \text{Ran}^G(X) \\
\text{Ran}(X) \times_{\text{Spec} \ k} \text{Ran}(X) \\
\text{Ran}(X) \times_{\text{Spec} \ k} \text{Ran}(X).
\]
Ignoring the distinction between !-sheaves and ∗-sheaves for the moment, we can think of this diagram as supplying a map
\[ \theta : Z_{\text{Ran}(X)} \boxtimes B \rightarrow A \boxtimes \omega_{\text{Ran}(X)} \]
For every pair \((S,T)\), we can pass to the stalk at \(S\) and costalk at \(T\) to obtain a map
\[ \theta_{S,T} = \bigotimes_{t \in T} C^*(BG_t; Z_t) \rightarrow \bigotimes_{s \in S} C^*(\text{Gr}_s^G; Z_t). \]
Geometrically, this map arises from a map of prestacks
\[ \rho_{S,T} : \prod_{s \in S} \text{Gr}_s^G \rightarrow \prod_{t \in T} BG_t. \]
Note that if \(S \neq T\), then this map exhibits some degenerate behavior. For example, if there exists an element \(s_0 \in S\) which does not belong to \(T\), then the map \(\rho_{S,T}\) factors through the product \(\prod_{s \neq s_0} \text{Gr}_s^G\), which we can think of as parametrizing \(G\)-bundles on \(X - \{s_0\}\) with a trivialization on \(X - S\) (in order to restrict a \(G\)-bundle to the set \(T\), we do not need it to be defined at the point \(s_0\)). Similarly, if there exists an element \(t_0 \in T\) which does not belong to \(S\), then the map \(\rho_{S,T}\) factors through \(\prod_{t \neq t_0} BG_t\) (since any \(G\)-bundle trivial on \(X - S\) will be trivial at the point \(t_0\)). In either case, we conclude that the composite map
\[ \bigotimes_{t \in T} C^*_\text{red}(BG_t; Z_t) \rightarrow \bigotimes_{t \in T} C^*(BG_t; Z_t), \]
\[ \theta_{S,T} \bigotimes_{s \in S} C^*(\text{Gr}_s^G; Z_t) \rightarrow \bigotimes_{s \in S} C^*_\text{red}(\text{Gr}_s^G; Z_t) \]
vanishes.

It is possible to introduce “reduced versions” of the sheaves \(A\) and \(B\), which we will denote by \(A_{\text{red}}\) and \(B_{\text{red}}\), whose (co)stalks are given by
\[ S^* A_{\text{red}} = \bigotimes_{s \in S} C^*(\text{Gr}_s^G; Z_t) \quad T^! B_{\text{red}} = \bigotimes_{t \in T} C^*(BG_t; Z_t). \]
An elaboration of the above argument shows that \(\theta\) induces a map
\[ \theta_{\text{red}} : Z_{\text{Ran}(X)} \boxtimes B_{\text{red}} \rightarrow A_{\text{red}} \boxtimes \omega_{\text{Ran}(X)} \]
which vanishes away from the diagonal of \(\text{Ran}(X) \times_{\text{Spec} k} \text{Ran}(X)\). Heuristically, this means that \(\theta_{\text{red}}\) factors through a map
\[ Z_{\text{Ran}(X)} \boxtimes B_{\text{red}} \rightarrow \Delta^! A_{\text{red}} \boxtimes \omega_{\text{Ran}(X)}, \]
which we can identify with a map
\[ B_{\text{red}} = \Delta^*(Z_{\text{Ran}(X)} \boxtimes B_{\text{red}}) \rightarrow \Delta^! (A_{\text{red}} \boxtimes \omega_{\text{Ran}(X)}) \simeq A_{\text{red}}. \]

The main idea of our proof is to show that this map is an equivalence. However, this does not quite make sense as we have formulated it: the right hand side is a !-sheaf on \(\text{Ran}(X)\), and the left hand side is a ∗-sheaf on \(\text{Ran}(X)\). Moreover, many of the objects which appeared in the above discussion (like the external tensor product \(A_{\text{red}} \boxtimes \omega_{\text{Ran}(X)}\)) need to be interpreted as some sort of hybrid between ∗-sheaves and !-sheaves. It will therefore be convenient to recast the above discussion in a less symmetrical way (essentially by “pushing forward” all of our sheaves onto the second copy of \(\text{Ran}(X)\)), which involves only !-sheaves on \(\text{Ran}(X)\).
Let us now dispense with heuristics and describe the strategy we will actually pursue. For every nonempty finite set $S$, we let $\text{Ran}_G(X)_S$ denote the fiber product $\text{Ran}_G(X) \times_{\text{Spec } k} X_S$. We then have maps of Ran-prestacks

$$\text{Ran}_G(X)_S \times_{\text{Spec } k} \text{Ran}(X) \to X_S \times_{\text{Spec } k} \text{Bun}_G(X) \times_{\text{Spec } k} \text{Ran}(X) \to \text{Ran}^G(X),$$

depending functorially on $S$. We therefore obtain maps of $\mathcal{I}$-sheaves

$$\mathcal{B} \to C^*(X_S; \mathbb{Z}_\ell) \otimes C^*(\text{Bun}_G(X); \mathbb{Z}_\ell) \otimes \omega_{\text{Ran}(X)} \to C^*(\text{Ran}_G(X)_S; \mathbb{Z}_\ell),$$

depending functorially on $S$. Passing to chiral homology, we obtain maps

$$\int_{\text{Ran}(X)} \mathcal{B} \to C^*(\text{Bun}_G(X); \mathbb{Z}_\ell).$$

The inverse limit of the maps $\beta_S$ as $S$ varies can be identified with the natural map $C^*(\text{Ran}(X) \times_{\text{Spec } k} \text{Bun}_G(X); \mathbb{Z}_\ell) \to C^*(\text{Ran}_G(X); \mathbb{Z}_\ell)$: this is predual to the equivalence

$$C_*(\text{Bun}_G(X); \mathbb{Z}_\ell) \cong C_*(\text{Ran}(X); \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} C_*(\text{Bun}_G(X); \mathbb{Z}_\ell) \cong C_*(\text{Ran}(X) \times_{\text{Spec } k} \text{Bun}_G(X); \mathbb{Z}_\ell) \to C_*(\text{Ran}_G(X); \mathbb{Z}_\ell),$$

supplied by nonabelian Poincare duality. The inverse limit of the maps $\alpha_S$ as $S$ varies can be identified with the map

$$\int_{\text{Ran}(X)} \mathcal{B} \to C^*(\text{Bun}_G(X); \mathbb{Z}_\ell)$$

that we discussed in the previous lecture. Consequently, we are reduced to proving the following:

**Proposition 1.** The induced map

$$\int_{\text{Ran}(X)} \mathcal{B} \to \varprojlim_{S} C^*(\text{Ran}_G(X)_S; \mathbb{Z}_\ell)$$

is an equivalence in $\text{Mod}_{\mathbb{Z}_\ell}$.

We will prove this by factoring the the composite map

$$\xi : \text{Ran}_G(X)_S \times_{\text{Spec } k} \text{Ran}(X) \to X_S \times_{\text{Spec } k} \text{Bun}_G(X) \times_{\text{Spec } k} \text{Ran}(X) \to \text{Ran}^G(X)$$

in a different way. For this, we need an auxiliary construction.

**Construction 2.** Fix a nonempty finite set $S$ and a pair of subsets $K_- \subseteq K_+ \subseteq S$. We define a prestack $\mathcal{C}(K_-, K_+)$ as follows:

- The objects of $\mathcal{C}(K_-, K_+)$ are tuples $(R, \mu, \nu : T \to X(R), \mathcal{P}, \gamma)$ where $R$ is a finitely generated $k$-algebra, $T$ is a nonempty finite set, $\mu : S \to X(R)$ and $\nu : T \to X(R)$ are maps of sets such that $|\mu(K_+)| \cap |\nu(T)| = \emptyset$, $\mathcal{P}$ is a $G$-bundle on $X_R - \mu(S_-)$ which can be extended to a $G$-bundle on $X_R$, and $\gamma$ is a trivialization of $\mathcal{P}$ over $X_R - |\mu(S)|$.

- A morphism from $(R, \mu, \nu : T \to X(R), \mathcal{P}, \gamma)$ to $(R', \mu', \nu' : T' \to X(R'), \mathcal{P}', \gamma')$ consists of an $k$-algebra homomorphisms $\phi : R \to R'$ such that $\mu'$ is given by the composition $S \xrightarrow{\mu_0} X(R) \xrightarrow{X(\phi)} X(R')$, a surjection of finite sets $\lambda : T \to T'$ which fits into a commutative diagram

$$\begin{array}{ccc}
T & \xrightarrow{\lambda} & T' \\
\downarrow \nu & & \downarrow \nu' \\
X(R) & \xrightarrow{X(\phi)} & X(R')
\end{array}$$

\[3\]
and a $G$-bundle isomorphisms between $\mathcal{P} \times \text{Spec } R \text{Spec } R'$ and $\mathcal{P}'$ over the scheme $X_{R'} - |\mu'(K_-)|$ which carries $\gamma$ to $\gamma'$.

**Remark 3.** If the set $S$ and the subset $K_+ \subseteq S$ are fixed, then we can regard $\mathcal{C}(K_-, K_+)$ as a covariant functor of $K_-$: for every inclusion $K_- \subseteq K'_- \subseteq K_+$, we have a forgetful functor

$$\mathcal{C}(K_-, K_+) \to \mathcal{C}(K'_-, K_+)$$

given by restriction of $G$-bundles. Here it is helpful to think of $\mathcal{C}(K'_-, K_+)$ as the quotient of $\mathcal{C}(K_-, K_+)$ obtained by identifying $G$-bundles which differ away from the image of $K'_+$ in $X$.

**Remark 4.** If the set $S$ and the subset $K_- \subseteq S$ are fixed, then we can regard $\mathcal{C}(K_-, K_+)$ as a contravariant functor of $K_-$: for every inclusion $K_- \subseteq K_+ \subseteq K'_+$, we can identify $\mathcal{C}(K_-, K'_+)$ with a full subcategory of $\mathcal{C}(K_-, K_+)$ (given by those objects $(R, \mu, \nu : T \to X(R), \mathcal{P}, \gamma)$ which satisfy the additional condition that $\mu(K'_+) \cap \nu(T) = \emptyset$).

**Example 5.** If $K_- = K_+ = \emptyset$, then we can identify $\mathcal{C}(K_-, K_+)$ with the product $\text{Ran}_G(X)_S \times \text{Spec } k \text{Ran}(X)$.

**Definition 6.** Fix a nonempty finite set $S$. We let $\text{Ran}_G(X)_S$ denote the category obtained via the Grothendieck construction on the functor $(K_-, K_+) \mapsto \mathcal{C}(K_-, K_+)$. More precisely, we have the following:

- The objects of $\text{Ran}_G(X)_S$ are tuples $(R, K_-, K_+, \mu, \nu : T \to X(R), \mathcal{P}, \gamma)$ where $R$ is a finitely generated $k$-algebra, $K_-$ and $K_+$ are subsets of $S$ with $K_- \subseteq K_+$, $T$ is a nonempty finite set, $\mu : S \to X(R)$ and $\nu : T \to X(R)$ are maps of sets such that $|\mu(K_+)| \cap |\nu(T)| = \emptyset$, $\mathcal{P}$ is a $G$-bundle on $X_R - |\mu(K_-)|$ which can be extended to a $G$-bundle on $X_R$ and $\gamma$ is a trivialization of $\mathcal{P}$ over $X_R - |\mu(S)|$.

- There are no morphisms

$$(R, K_-, K_+, \mu, \nu : T \to X(R), \mathcal{P}, \gamma) \to (R', K'_-, K'_+, \mu', \nu' : T' \to X(R'), \mathcal{P}', \gamma')$$

unless $K'_- \subseteq K_- \subseteq K_+ \subseteq K'_+$. If this condition is satisfied, then a morphism from $(R, K_-, K_+, \mu, \nu : T \to X(R), \mathcal{P}, \gamma)$ to $(R', K'_-, K'_+, \mu', \nu' : T' \to X(R'), \mathcal{P}', \gamma')$ consists of a $k$-algebra homomorphism $\phi : R \to R'$ carrying $\mu$ to $\mu'$, a surjection of finite sets $\lambda : T \to T'$ which fits into a commutative diagram

$$
\begin{array}{ccc}
T & \xrightarrow{\lambda} & T' \\
\downarrow{\nu} & & \downarrow{\nu'} \\
X(R) & \xrightarrow{X(\phi)} & X(R')
\end{array}
$$

and a $G$-bundle isomorphism between $\mathcal{P}$ and $\mathcal{P}'$ over the scheme $X_{R'} - |\mu'(K'_-)|$ which carries $\gamma$ to $\gamma'$.

The construction $(R, K_-, K_+, \mu, \nu : T \to X(R), \mathcal{P}, \gamma) \mapsto (R, T, \nu, \mathcal{P} |_{|\nu(T)|})$ determines a forgetful functor $f_S : \text{Ran}_G(X)_S \to \text{Ran}_G(X)$. We let $\mathcal{B}_S$ denote the lax !-sheaf on $\text{Ran}(X)$ given by the formula

$$\mathcal{B}_S(T) = [\text{Ran}_G(X)_S \times \text{Ran}_G(X)]_{X_T}.$$

Note that the map $f_S$ induces a map of lax !-sheaves $\mathcal{B} \to \mathcal{B}_S$, depending functorially on $S$. Moreover, the identification $\mathcal{C}(\emptyset, \emptyset) \simeq \text{Ran}_G(X)_S \times \text{Spec } k \text{Ran}(X)$ determines a fully faithful embedding

$$\text{Ran}_G(X)_S \times \text{Spec } k \text{Ran}(X) \hookrightarrow \text{Ran}_G(X)_S,$$
which induces a pullback map $\mathcal{B}_S \to C^\ast(Ran_G(X)_S; \mathbb{Z}_\ell) \otimes \omega_{\text{Ran}(X)}$. Using the commutativity of the diagram

$$
\begin{array}{ccc}
\text{Ran}_G(X)_S \times_{\text{Spec } k} \text{Ran}(X) & \longrightarrow & \text{Bun}_G(X) \times_{\text{Spec } k} \text{Ran}(X) \\
\downarrow & & \downarrow \\
\text{Ran}_G(X)_S & \longrightarrow & \text{Ran}_G(X) 
\end{array}
$$

we see that the map $\xi$ of Proposition 1 can be identified with the composition

$$
\begin{aligned}
\int_{\text{Ran}(X)} \mathcal{B} & \xrightarrow{\xi'} \int_{\text{Ran}(X)} \lim_S \mathcal{B}_S \\
& \xrightarrow{\xi''} \lim_S \int_{\text{Ran}(X)} (C^\ast(Ran_G(X)_S; \mathbb{Z}_\ell) \otimes \omega_{\text{Ran}(X)}) \\
& \simeq \lim_S C^\ast(Ran_G(X)_S; \mathbb{Z}_\ell).
\end{aligned}
$$

We are therefore reduced to proving the following pair of assertions:

**Proposition 7.** The map $\xi''$ is an equivalence in $\text{Mod}_{\mathbb{Z}_\ell}$.

**Proposition 8.** The canonical map $\mathcal{B} \to \lim_S \mathcal{B}_S$ is an equivalence of $!$-sheaves on $\text{Ran}(X)$.

The proof of Proposition 7 is mostly formal: the difficulty lies in showing that passage to the inverse limit over $S$ "commutes" with passage to chiral homology. In terms of our heuristic picture, this is because the $\ast$-sheaf $\mathcal{A}_\text{red}$ is generated by compactly supported sections: in fact, in any given degree, the cohomologies of the sheaf $\mathcal{A}_\text{red}$ are supported on the substack $\text{Ran}(X) \leq n$ for $n \gg 0$. We will not present the details in class.

Proposition 8 can be regarded as a local calculation on the Ran space, which relates the cohomology of the Grassmannians $\text{Gr}_{G,x}$ to the cohomology of the classifying stacks $BG_y$. We will return to this in the next lecture.