Koszul Duality (Lecture 23)

April 2, 2014

In this lecture, we will continue our discussion of $\mathbb{E}_n$-algebras over a commutative ring R.

Definition 1. Let A be an $\mathbb{E}_n$-algebra over R. An *augmentation* on A is a map of $\mathbb{E}_n$-algebras $\epsilon : A \to R$.

The theory of augmented $\mathbb{E}_n$-algebras behaves as one would expect from classical algebra. If $\epsilon : A \to R$ is an augmentation, we let $\mathfrak{m}_A$ denote the fiber of ϵ, which we refer to as the *augmentation ideal* of A. Then $\mathfrak{m}_A$ inherits the structure of a nonunital $\mathbb{E}_n$-algebra. Moreover, the construction

$$(\epsilon : A \to R) \mapsto \mathfrak{m}_A$$

induces an equivalence from the ∞-category of augmented $\mathbb{E}_n$-algebras over R to the ∞-category of nonunital $\mathbb{E}_n$-algebras over R.

Definition 2. Let A and B be augmented $\mathbb{E}_n$-algebras over R. A *pairing* between A and B is an augmentation on the tensor product

$$\epsilon : A \otimes B \to R$$

which restricts to the given augmentations on A and B.

Proposition 3. *Let A be an augmented $\mathbb{E}_n$-algebra over R. Then there exists a pairing $\epsilon : A \otimes B \to R$ with the following universal property: for every augmented $\mathbb{E}_n$-algebra B', the data of a morphism of augmented $\mathbb{E}_n$-algebra $B' \to B$ is equivalent to the data of a pairing $\epsilon' : A \otimes B' \to R$.*

In the situation of Proposition 3, the augmented $\mathbb{E}_n$-algebra B is functorially determined by A. We will refer to it as the *Koszul dual* of A.

Example 4. When $n = 0$, the ∞-category of nonunital $\mathbb{E}_n$-algebras in Mod_R is equivalent to Mod_R itself. Consequently, every augmented $\mathbb{E}_0$-algebra A over R can be written uniquely as a direct sum $R \oplus \mathfrak{m}_A$, where $\mathfrak{m}_A$ is an arbitrary object of Mod_R. If B is another augmented $\mathbb{E}_0$-algebra over R, then a pairing of A with B is a map

$$\epsilon : (R \oplus \mathfrak{m}_A) \otimes (R \oplus \mathfrak{m}_B) \to R$$

which is the identity on $R \otimes R$ and vanishes on $\mathfrak{m}_A \otimes R$ and $R \otimes \mathfrak{m}_B$. Consequently, the data of a pairing is equivalent to the data of an R-linear map

$$\mathfrak{m}_A \otimes \mathfrak{m}_B \to R,$$

or equivalently of a map $\mathfrak{m}_B \to \mathfrak{m}_A^\vee$. It follows that the Koszul dual of $A = R \oplus \mathfrak{m}_A$ is the augmented $\mathbb{E}_0$-algebra $R \oplus \mathfrak{m}_A^\vee$.

Example 5. Let A be an augmented $\mathbb{E}_1$-algebra over R. Then we can regard R as an A-module via the augmentation $A \to R$. A pairing $A \otimes B \to R$ can be identified with an action of B on R which commutes with the given action of A on R. Such actions are classified by maps from B into the (derived) endomorphism algebra $\mathrm{End}_A(R)$. Consequently, we can identify the Koszul dual of A with the endomorphism algebra $\mathrm{End}_A(R)$, with augmentation given by the forgetful map $\mathrm{End}_A(R) \to \mathrm{End}_R(R)$.

Remark 6. Let A be an augmented $\mathbb{E}_1$-algebra over R. Then its Koszul dual is given by

$$\begin{aligned}\mathrm{End}_A(R) &\simeq \mathrm{Hom}_A(R,R)\\ &\simeq \mathrm{Hom}_R(R\otimes_A R, R)\\ &= (R\otimes_A R)^\vee.\end{aligned}$$

In other words, it is given by the R-linear dual of the relative tensor product $R\otimes_A R$.

The construction $A\mapsto R\otimes_A R$ is called the *bar construction*; it carries augmented algebra objects of Mod_R to augmented coalgebra objects of Mod_R. Moreover, the bar construction is compatible with tensor products. Consequently, if A is an augmented $\mathbb{E}_n$-algebra over R, we can apply the bar construction n times to obtain an $\mathbb{E}_n$-coalgebra over R, whose R-linear dual is the Koszul dual of A.

Example 7. Let (Y,y) be a pointed space. Then we can regard $C_*(\Omega^n Y;R)$ and $C^*(Y;R)$ as augmented $\mathbb{E}_n$-algebras over R (with augmentations given by the maps $Y\to *$ and $\{y\}\hookrightarrow Y$, respectively). If Y is $(n-1)$-connected, then one can show that the $\mathbb{E}_n$-algebra $C^*(Y;R)$ is Koszul dual to $C_*(\Omega^n Y;R)$. It is not always true that $C_*(\Omega^n Y;R)$ can be recovered as the Koszul dual of $C^*(Y;R)$; this requires some additional finiteness assumptions.

The relevance of Koszul duality to our situation stems from the following:

Proposition 8. *Let A be an augmented $\mathbb{E}_n$-algebra over R and let B be its Koszul dual, so that the augmentation ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ can be regarded as nonunital $\mathbb{E}_n$-algebras over R. Then the cosheaf $\mathcal{F}_{\mathfrak{m}_B}$ is Verdier dual to $\mathcal{F}_{\mathfrak{m}_A}$ on the Ran space $\mathrm{Ran}(\mathbb{R}^n)$. More precisely, if we let $\mathcal{F}^\vee_{\mathfrak{m}_A}$ denote the sheaf given by*

$$\mathcal{F}^\vee_{\mathfrak{m}_A}(U) = (\mathcal{F}_{\mathfrak{m}_A}(U))^\vee,$$

then $\mathcal{F}_{\mathfrak{m}_B}$ is the cosheaf given by $(\mathcal{F}^\vee_{\mathfrak{m}_A})_c$ (see Lecture 21).

Proposition 8 is extremely relevant to the subject matter of this course:

Example 9. Let G_0 be a split reductive group over the field $\mathbf{C}$ of complex numbers, let X be an algebraic curve over $\mathbf{C}$, and set $G = X\times_{\mathrm{Spec}\,\mathbf{C}} G_0$. If $x\in X$ is a point and $t\in\mathcal{O}_x$ is a local coordinate at x, then the $\mathbf{C}$-valued points of the affine Grassmannian $\mathrm{Gr}_{G,x}$ can be regarded as a topological space $G_0(\mathbf{C}((t)))/G_0(\mathbf{C}[[t]])$. Here $G_0(\mathbf{C}((t)))$ has the homotopy type of the free loop space $\mathrm{Map}(S^1, G_0(\mathbf{C}))$, and $G_0(\mathbf{C}[[t]])$ has the homotopy type of $G_0(\mathbf{C})$, so that $\mathrm{Gr}_{G,x}$ has the homotopy type of the based loop space $\Omega G_0(\mathbf{C})\simeq \Omega^2(BG_0(\mathbf{C}))$. It follows that we can regard $A = C_*(\mathrm{Gr}_{G,x};\mathbf{Q})$ as an $\mathbb{E}_2$-algebra. Let $\mathcal{F}_A$ denote the associated cosheaf on $\mathrm{Ran}(\mathbb{R}^2)$, and let $\mathcal{F}^\vee_A$ be the dual sheaf. For any open embedding $\mathbb{R}^2\to X$, we can identify $\mathcal{F}^\vee_A$ with the restriction of the sheaf $\mathcal{A}\in\mathrm{Shv}(\mathrm{Ran}(X))$ to $\mathrm{Ran}(\mathbb{R}^2)$, where $\mathcal{A}$ is the direct image of the constant sheaf $\mathbf{Q}$ along the projection map $\mathrm{Ran}_G(X)\to\mathrm{Ran}(X)$.

Similarly, we can regard $C^*(\mathrm{BG}_0;\mathbf{Q})$ as an $\mathbb{E}_2$-algebra B over $\mathbf{Q}$, so that we have an associated cosheaf $\mathcal{F}_B$ on $\mathrm{Ran}(\mathbb{R}^2)$. Let us abuse notation by identifying $\mathcal{F}_B$ with the associated !-sheaf (under the equivalence $\mathrm{cShv}(\mathrm{Ran}(\mathbb{R}^2))\simeq \mathrm{Shv}^!(\mathrm{Ran}(\mathbb{R}^2))$ supplied by covariant Verdier duality). This construction also "globalizes": that is, we can identify $\mathcal{F}_B$ with the restriction of a !-sheaf $\mathcal{B}\in\mathrm{Shv}^!(\mathrm{Ran}(X))$, whose costalk at a point $x\in X$ is given by $C^*(\mathrm{BG}_0;\mathbf{Q})$ (we introduced an algebro-geometric version of $\mathcal{B}$ in Lecture 20).

The algrebas A and B are naturally augmented, with augmentation ideals

$$\mathfrak{m}_A = C^{\mathrm{red}}_*(\mathrm{Gr}_{G,x};\mathbf{Q}) \qquad \mathfrak{m}_B = C^*_{\mathrm{red}}(\mathrm{BG}_0;\mathbf{Q}).$$

To these nonunital $\mathbb{E}_2$-algebras, we can associate a sheaf $\mathcal{A}^{\mathrm{red}}$ and a !-sheaf $\mathcal{B}^{\mathrm{red}}$ on $\mathrm{Ran}(X)$, given by "reduced" versions of the constructions above. Using Example 7, we see that B is the Koszul dual of A. It follows from Proposition 8 that we can identify $\mathcal{B}^{\mathrm{red}}$ with the !-sheaf obtained by extracting compactly supported sections of $\mathcal{A}^{\mathrm{red}}$. We therefore expect a close relationship between sections of $\mathcal{A}$ and compactly supported sections of $\mathcal{B}$, which we will describe in the next lecture.