Existence of Borel Reductions II (Lecture 15)

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Throughout this lecture, we let $k$ be an algebraically closed field, $X$ an algebraic curve over $k$, $G$ a smooth affine group scheme over $X$, $G_0$ the generic fiber of $G$, $B_0 \subseteq B$ the scheme-theoretic closure of $B_0$ in $G$. Let $\mathcal{P}$ be a $G$-bundle on $X$ and let $\pi : \mathcal{P}/B \to X$ be the projection map. Our goal is to prove the following result which was needed in the previous lecture:

**Theorem 1.** Let $\mathcal{P}$ be a $G$-bundle on $X$. Then there exists a section $s$ of the projection map $\pi : \mathcal{P}/B \to X$ such that $H^1(X; s^*T_\pi) \simeq 0$.

We will prove Theorem 1 under the assumption that the generic fiber $G_0$ is split reductive. The statement also holds under the assumption that $G_0$ is semisimple and simply connected, but requires a more complicated argument.

Let $G'$ be the unique split reductive algebraic group over $k$ such that there is an isomorphism $\alpha : \text{Spec} K_X \times_{\text{Spec} k} G' \simeq G_0$ and let $B'$ be a Borel subgroup of $G'$. Since all Borel subgroups of $G_0$ are conjugate, we may assume without loss of generality that the isomorphism $\alpha$ carries $\text{Spec} K_X \times_{\text{Spec} k} B'$ to $B_0$. We may therefore choose a dense open subset $U \subseteq X$ such that $\alpha$ extends to an isomorphism $G' \times_{\text{Spec} k} U \simeq G \times_X U$ carrying $B' \times_{\text{Spec} k} U$ to $B \times_X U$.

In the last lecture, we showed that $\mathcal{P}$ admits a $B$-reduction, which we can identify with a section $s_0$ of the projection map $\pi : \mathcal{P}/B \to X$. Let $\mathcal{Q}$ denote the associated $B$-bundle on $X$, so that $\mathcal{Q}$ is trivial at the generic point of $X$. Shrinking the open set $U$, we may assume that $\mathcal{Q}|_U$ is trivial. It follows that $\alpha$ determines an isomorphism
\[
\beta : G'/B' \times_{\text{Spec} k} U \simeq \mathcal{P}/B \times_X U,
\]
which carries the “zero section” of the projection map $G'/B' \times_{\text{Spec} k} U \to U$ to the map $s_0|_U$.

We would now like to extend $\beta$ to a map
\[
\overline{\beta} : G'/B' \times_{\text{Spec} k} X \to \mathcal{P}/B.
\]
Unfortunately, such an extension need not exist. However, we can always find such an extension after suitably “blowing-up” the variety $G'/B' \times_{\text{Spec} k} X$.

**Construction 2** (Dilation). Let $Y$ be a quasi-projective $k$-scheme equipped with a smooth map $f : Y \to X$, and let $y \in Y(k)$ be a point having image $x \in X(k)$. Let $\mathcal{I}_y$ denote the ideal sheaf of $y$ in $Y$. Let $A_y$ denote the direct limit
\[
\lim_{\rightarrow} \mathcal{I}_y^m \otimes_{O_Y} f^* O_X(mx).
\]
Then $A_y$ is a quasi-coherent sheaf of algebras on $Y$, which determines a map of affine schemes $D_y(Y) \to Y$.

We will refer to $D_y(Y)$ as the dilation of $Y$ at the point $y$.

**Remark 3.** We can describe $D_y(Y)$ as the scheme obtained by first blowing up $Y$ at the point $y$, and then removing the closed subscheme obtained by blowing up $Y \times_X \{x\}$ at the point $y$.

**Remark 4.** Suppose that $Y$ is smooth over $X$. Then $D_y(Y)$ is also smooth over $X$. Moreover, if $g : D_y(Y) \to Y$ denotes the projection map, then we have a canonical isomorphism $T_{D_y(Y)/X} \simeq g^* T_Y/X(\{x\})$. 
If \( Y = X \), then \( D_y(Y) \simeq Y \) for any point \( y \in Y(k) \). By functoriality, we see that if \( s : X \to Y \) is a section of the projection map \( f : Y \to X \) which passes through the point \( Y \), then \( s \) lifts (uniquely!) to a section \( \tilde{s} : X \to D_y(Y) \) of the projection map \( D_y(Y) \to X \).

Suppose that \( f_0 : Y_0 \to X \) is a map equipped with a section \( s_0 \), and that we are given a finite sequence of points \( x_1, \ldots, x_m \in X(k) \) (which need not be distinct). We can then define sequence of \( X \)-schemes \( f_i : Y_i \to X \) and section \( s_i : X \to Y_i \) by the formula \( Y_i = D_{s_{i-1}}(x_i)Y \), with \( s_i \) the unique lift of \( s_{i-1} \). The scheme \( Y_m \) depends only on the divisor \( D = x_1 + \cdots + x_m \) and the section \( s_0 \). In this case, we will say that \( Y_m \) is obtained from \( Y \) by dilatation along \( s_0(D) \).

**Warning 5.** This is an abuse of terminology: the scheme \( Y_m \) depends not only on the set \( s_0(D) \), but also on the section \( s_0 \) and the divisor \( D \).

**Remark 6.** In the situation above, suppose we are given another section \( s : X \to Y_0 \) of the map \( f_0 \). If the sections \( s \) and \( s' \) agree on the divisor \( D \), then \( s \) can be lifted to a map \( \tilde{s} : X \to Y_m \).

We will need the following algebra-geometric fact:

**Proposition 7.** Let \( f : Y \to X \) be a map of integral \( k \)-schemes equipped with a section \( h \), let \( Z \) be a quasi-projective \( k \)-scheme, let \( U \subseteq X \) be a dense open set, and suppose that we are given a map \( \beta : U \times_X Y \to Z \) such that \( \beta \circ h|_U \) can be extended to a map \( X \to Z \).

Then there exists an effective divisor \( D \subseteq X \) supported in \( X \setminus U \) such that \( \beta \) factors as a composition

\[
U \times_X Y \hookrightarrow M \xrightarrow{\tilde{\beta}} Z,
\]

where \( M \) denotes the scheme obtained from \( Y \) by dilatation along \( h(D) \).

Let us now apply Proposition 7 to the case where \( Y = G'/B' \times_{\text{Spec} k} X \), \( Z = \mathcal{P}/B \), \( \beta \) is our isomorphism \( G'/B' \times_{\text{Spec} k} U \simeq \mathcal{P}/B \times_X U \), and \( h \) is the zero section of the projection map \( G'/B' \times_{\text{Spec} k} X \to X \). It follows that there exists an effective divisor \( D \subseteq X \) supported in \( X \setminus U \) and a commutative diagram

\[
\begin{array}{ccc}
\phi & \rightarrow & M \\
G'/B' \times_{\text{Spec} k} X & \xrightarrow{\beta} & \mathcal{P}/B, \\
\end{array}
\]

where \( M \) is obtained from \( (G'/B') \times_{\text{Spec} k} X \) by dilatation along \( h(D) \).

Let \( h' \) be any section of the projection map \( (G'/B') \times_{\text{Spec} k} X \) which agrees with \( h \) on the divisor \( D \). Then \( h' \) lifts (uniquely) to a map \( \overline{h}' : X \to M \), so \( \overline{\beta} \circ \overline{h}' : X \to \mathcal{P}/B \) determines a \( B \)-reduction of \( \mathcal{P} \). Moreover, we have a map of vector bundles

\[
\overline{h}'^* T_{M/X} \to (\overline{\beta} h')^* T_\pi
\]

on \( X \) which is an isomorphism over the open set \( U \), and therefore induces a surjection

\[
H^1(X; \overline{h}'^* T_{M/X}) \to H^1(X; (\overline{\beta} h')^* T_\pi).
\]

Let \( \psi : (G'/B') \times_{\text{Spec} k} X \to X \) denote the projection map, and let \( T_\psi \) denote the relative tangent bundle of \( \psi \). Applying Remark 4 repeatedly, we obtain an isomorphism \( T_{M/X} \simeq T_\psi(-D) \), so that

\[
H^1(X; \overline{h}'^* T_{M/X}) \simeq H^1(X; (h'^* T_\psi)(-D)).
\]

Note that \( h' \) can be identified with a map \( g : X \to G'/B' \), and \( h'^* T_\psi \) with the pullback \( g^* T_{G'/B'} \), where \( T_{G'/B'} \) denotes the tangent bundle to the flag variety \( G'/B' \). To complete the proof of Theorem 1, it will suffice to prove the following:
Theorem 8. Let $D$ be an arbitrary effective divisor in $X$. Then there exists a map $g : X \to G'/B'$ such that $g|_D$ is constant and the cohomology group $H^1(X; g^*T_{(G'/B')}(-D))$ vanishes.

Fix a maximal torus $T' \subseteq B'$. Let $\Lambda^* = \text{Hom}(T', G_m)$ denote the character lattice of $T_0$, and let $\Lambda_* = \text{Hom}(G_m, T')$ be its cocharacter lattice. Every element $\lambda \in \Lambda^*$ determines a group homomorphism $B' \to G_m$, which determines an equivariant line bundle $\mathcal{L}_\lambda$ on the flag variety $G'/B'$. If $g : X \to G'/B'$ is an arbitrary map, then the function $\lambda \mapsto \deg(g^*\mathcal{L}_\lambda)$ can be regarded as an additive map from $X^*(T_0)$ to $\mathbb{Z}$, which we can identify with an element of $X_*(T_0)$. We will refer to element as the degree of $g$ and denote it by $\deg(g)$.

Let $\Phi_-$ denote the set of negative roots of $G'$ with respect to $(B', T')$: that is, the subset of $\Lambda^*$ consisting of characters which appear as roots of $G'$ but not of the Borel subgroup $B'$. Unwinding the definitions, we see that the tangent bundle $T_{G'/B'}$ admits a filtration whose successive quotients are the line bundles $\{\mathcal{L}_\lambda\}_{\lambda \in \Phi_-}$. Consequently, to prove the vanishing of $H^1(X; g^*T_{(G'/B')}(-D))$, it will suffice to prove the vanishing of $H^1(X; (g^*\mathcal{L}_\lambda)(-D))$ for $\lambda \in \Phi_-$. By the Riemann-Roch theorem, this vanishing is automatic provided that $\deg(\mathcal{L}_\lambda) > 2g - 2 + d$, where $g$ is the genus of the curve $X$ and $d$ is the degree of the divisor $D$. We are therefore reduced to proving the following:

Theorem 9. Let $D$ be an arbitrary effective divisor in $X$ and let $n$ be a positive integer. Then there exists a map $g : X \to G'/B'$ such that $g|_D$ vanishes, and $(\deg(g), \lambda) \geq n$ for $\lambda \in \Phi_-$. 

Choose any map $X \to \mathbb{P}^1$ which has degree $\geq n$ and is constant on the divisor $D$. Then any composite map $X \to \mathbb{P}^1 \xrightarrow{\phi} G'/B'$ is constant on $D$ and can therefore (after translating by a point of $G'$) be assumed to vanish on $D$. We are therefore reduced to proving:

Theorem 10. There exists a map $g : \mathbb{P}^1 \to G'/B'$ such that $\deg(g)$ is strictly antidominant: that is, $\langle \deg(g), \lambda \rangle > 0$ for $\lambda \in \Phi_-$. 

Example 11. When $G' = \text{SL}_2$, Theorem 10 asserts that there exists a map from $\mathbb{P}^1$ to itself of positive degree.

Example 12. Let $V$ be the standard representation of $\text{SL}_2$. Then $\text{Sym}^{n-1}(V)$ is an $n$-dimensional representation of $\text{SL}_2$, given by a map $\text{SL}_2 \to \text{SL}_n$. This map carries a Borel subgroup of $\text{SL}_2$ into a Borel subgroup of $G' = \text{SL}_n$, and therefore induces a map of flag varieties $g : \mathbb{P}^1 \to G'/B'$. An easy calculation shows that $\langle \deg(g), \lambda \rangle = 2$ for each negative simple root $\lambda$ of $G'$, so that $g$ satisfies the requirements of Theorem 10.

If the field $k$ has characteristic zero, then the argument of Example 12 can be generalized by consider a “principal $\text{SL}_2$” in the group $G'$. For a general argument which works in positive characteristic, we refer the reader to [1].

References


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