Existence of Borel Reductions I (Lecture 14)

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Throughout this lecture, we let \( k \) be an algebraically closed field, \( X \) an algebraic curve over \( k \), \( G \) a smooth affine group scheme over \( X \) whose generic fiber \( G_0 \) is semisimple and simply connected, \( B_0 \) a Borel subgroup of \( G_0 \), and \( B \) the scheme-theoretic closure of \( B_0 \) in \( G \). Our goal is to prove the following version of a theorem of Drinfeld and Simpson:

**Theorem 1.** Let \( R \) be a finitely generated \( k \)-algebra and let \( \mathcal{P} \) be a \( G \)-bundle on \( X_R \). Then, étale locally on \( \Spec R \), the \( G \)-bundle \( \mathcal{P} \) admits a \( B \)-reduction.

We begin by treating the case where \( R = k \). Our starting point is the following:

**Lemma 2.** Let \( \mathcal{P} \) be a \( G \)-bundle on \( X \) and let \( S \) be a finite set of closed points of \( X \). Then there exists an open set \( U \subseteq X \) containing \( S \) such that \( \mathcal{P}|_U \) is trivial.

**Proof.** We first recall that the fraction field \( \mathcal{K}_X \) is a field of dimension 1. It follows that any \( G \)-bundle on \( X \) is automatically trivial at the generic point of \( X \). In particular, if \( \mathcal{P} \) is a \( G \)-bundle on \( X \), then we can choose a trivialization of \( \mathcal{P} \) at the generic point of \( X \). Let us view this trivialization as a map \( \eta : \Spec K_X \) fitting into a commutative diagram

\[
\begin{array}{ccc}
\Spec K_X & \xrightarrow{\eta} & \mathcal{P} \\
\downarrow & & \downarrow \\
X. & & \\
\end{array}
\]

It follows that \( \eta \) can be extended to a map of \( X \)-schemes \( U \to \mathcal{P} \), where \( U \) is a dense open subset of \( X \), which we can assume is chosen to be as large as possible. We wish to show that after modifying the trivialization \( \eta \), we can arrange that \( S \subseteq U \).

Write \( S = \{ x_1, \ldots, x_n \} \). Since \( k \) is algebraically closed, the \( G \)-bundle \( \mathcal{P} \) is trivial at the the residue field of each of the points \( x_i \). Since \( G \) is smooth, we can extend these trivializations to maps \( \eta_i : \Spec \mathcal{O}_{x_i} \to \mathcal{P} \), where \( \mathcal{O}_{x_i} \) denotes the complete local ring of the curve \( X \) at the point \( x_i \) (so that \( \mathcal{O}_{x_i} \) is noncanonically isomorphic to a power series ring \( k[[t]] \)).

For \( 1 \leq i \leq n \), let \( K_{x_i} \) denote the fraction field of \( \mathcal{O}_{x_i} \), so that \( \eta \) and \( \eta_i \) determine two different trivializations of \( \mathcal{P}|_{\Spec K_{x_i}} \). These trivializations differ by some elements \( g_i \in G(K_{x_i}) \). Note that \( g_i \) belongs to the subgroup \( G(\mathcal{O}_{x_i}) \subseteq G(K_{x_i}) \) if and only if it is possibly to adjust the trivialization \( \eta_i \) to be compatible with the generic trivialization \( \eta \): that is, if and only if the point \( x_i \) is contained in \( U \).

To complete the proof, we wish to show that we can change the trivialization \( \eta \) to arrange that each \( g_i \) belongs to \( G(\mathcal{O}_{x_i}) \). In other words, we wish to prove that we can choose \( g \in G(K_X) \) so that each of the products \( gg_i \in G(K_{x_i}) \) belongs to \( G(\mathcal{O}_{x_i}) \).

Each of the fields \( K_{x_i} \) admits a topology, with a neighborhood basis of the identity element given by nonzero ideals in the discrete valuation ring \( \mathcal{O}_{x_i} \). This determines a topology on each of the groups \( G(K_{x_i}) \) and therefore also on the product \( \prod_{1 \leq i \leq n} G(K_{x_i}) \). By construction, the product \( \prod_{1 \leq i \leq n} G(\mathcal{O}_{x_i})g_i^{-1} \) is a nonempty open subset of \( \prod_{1 \leq i \leq n} G(K_{x_i}) \). It will therefore suffice to prove the following:

(\#) The map \( G(K_X) \to \prod_{1 \leq i \leq n} G(K_{x_i}) \) has dense image.
Note that assertion (\(\ast\)) depends only on the generic fiber \(G_0\) of \(G\). Choose a Borel subgroup \(B_0^-\) which is opposite to \(B_0\), so that the intersection \(T_0 = B_0 \cap B_0^-\) is a maximal torus in \(G_0\). Let \(U_0\) and \(U_0^-\) denote the unipotent radicals of \(B_0\) and \(B_0^-\), respectively. Then the Bruhat decomposition for \(G\) implies that the multiplication induces an open immersion

\[
U_0^- \times T_0 \times U_0 \hookrightarrow G_0
\]

whose image is a Zariski-dense open set \(V \subseteq G_0\). We need the following:

**Lemma 3.** Let \(R\) be a complete discrete valuation ring with maximal ideal \(m\), let \(K\) be the fraction field of \(R\), let \(Y\) be a smooth affine \(K\)-scheme, and let \(U \subseteq Y\) be a dense open set. Then \(U(K)\) is dense in \(Y(K)\) (where we equip \(Y(K)\) with the \(m\)-adic topology).

It follows from Lemma 3 that each \(V(K_{x_i})\) is dense in \(G_0(K_{x_i})\). It will therefore suffice to show that the map \(V(K_X) \to \bigoplus_{1 \leq i \leq n} V(K_{x_i})\) has dense image. Note that \(V\) factors (as a \(K_X\)-scheme) as a product of finitely many copies of \(G_0\), and the maximal torus \(T_0\). Moreover, as we saw in the previous lecture, the torus \(T_0\) is induced: that is, it can be written as a product \(\prod_{1 \leq j \leq m} \text{Res}^{L_j} K_X G_m\), where \(\{L_j\}_{1 \leq j \leq m}\) is a finite collection of separable extensions of \(K_X\). We are therefore reduced to proving that the maps

\[
K_X \to \bigoplus_{1 \leq i \leq n} K_{x_i}
\]

\[
L_j^\times \to \bigoplus_{1 \leq i \leq n} (K_{x_i} \otimes_{K_X} L_j)^\times
\]

have dense image, which we leave to the reader. \(\Box\)

**Proof of Lemma 3.** The assertion is local with respect to the Zariski topology on \(Y\). We may therefore assume without loss of generality that there exists an étale morphism of \(k\)-schemes \(\phi : Y \to \mathbb{A}^d\), where \(d\) is the dimension of \(Y\). Let \(Z\) denote the complement of \(U\) in \(Y\). Since \(U\) is dense, we have \(\dim(Z) < d\), so that the image under \(\phi\) of \(Z\) is contained in a proper closed subscheme of \(\mathbb{A}^d\). We may therefore choose a nonzero polynomial \(f(x_1, \ldots, x_d)\) which vanishes on the points of \(\phi(Z(K))\), so that \(\phi(Z(K))\) cannot contain any nonempty open subset of \(K^n\). It follows from Hensel’s lemma that \(\phi\) induces an open map \(Y(K) \to K^d\), so that \(Z(K)\) cannot contain any open subset of \(Y(K)\) and therefore \(U(K)\) is dense in \(Y(K)\), as desired. \(\Box\)

We now return to the proof of Theorem 1 in the special case where \(R = k\). Let \(\mathcal{P}\) be a \(G\)-bundle on \(X\). Then \(\mathcal{P}\) is equipped with a free action of \(G\) (in the category of \(X\)-schemes), and in particular a free action of \(B\). We let \(\mathcal{P}/B\) denote the quotient of \(\mathcal{P}\) by the action of \(B\). We have a canonical map \(\pi : \mathcal{P}/B \to X\), and we can identify \(B\)-reductions of \(\mathcal{P}\) with sections of the map \(\pi\).

**Remark 4.** By the quotient \(\mathcal{P}/B\), we refer to the quotient of \(\mathcal{P}\) by \(B\) in the category of fppf sheaves on \(X\). It follows from a general theorem of Artin that such a quotient is always representable by an algebraic space ([?]). In fact, one can show that \(\mathcal{P}/B\) is representable by a scheme, but this is not really important.

The generic fiber of \(\pi\) can be identified with the quotient \(G_0/B_0\) in the category of \(K_X\)-schemes. Since \(B_0\) was defined to be a Borel subgroup of \(G_0\), the quotient \(G_0/B_0\) is proper over \(\text{Spec} K_X\). It follows that the map \(\pi : \mathcal{P}/B \to X\) is generically proper: that is, it induces a proper map \((\mathcal{P}/B) \times_X W \to W\) for some dense open subset \(W \subseteq X\). The complement \(X - W\) consists of finitely many closed points \(x_1, \ldots, x_n \in X\). Applying Lemma 2, we can choose an open subset \(U \subseteq X\) containing each \(x_i\) such that \(\mathcal{P}|_U\) is trivial. In particular, the \(G\)-bundle \(\mathcal{P}|_U\) admits a reduction to \(B\), classified by a map \(s : U \to \mathcal{P}/B\) which is a section of \(\pi\). Since every point of \(X - U\) belongs to \(W\), the map \(s\) extends uniquely to a map \(\overline{s} : X \to \mathcal{P}/B\) using the valuative criterion for properness, which we can identify with a \(B\)-reduction of \(\mathcal{P}\). This completes the proof of Theorem 1 in the special case \(R = k\).

Let us now turn to the general case. Let \(R\) be a finitely generated \(k\)-algebra and let \(\mathcal{P}\) be a \(G\)-bundle on \(X_R\). As before, we let \(\mathcal{P}/B\) denote the quotient of \(\mathcal{P}\) by the action of \(B\). Let \(\text{Fl}^*\) be the \(R\)-scheme obtained
by Weil restriction of $\mathcal{P}/B$ along the projection map $X_R \to \text{Spec } R$ (see [2] for a careful discussion). In other words, $\text{Fl}$ is the $R$-scheme whose set of $A$-valued points $\text{Fl}(A)$ can be identified with the set of commutative diagrams

$$
\begin{array}{ccc}
X_A & \rightarrow & \mathcal{P}/B \\
\downarrow & & \downarrow \\
X_R.
\end{array}
$$

Unwinding the definitions, we see that there is a bijective correspondence between $\text{Fl}(A)$ and the set of isomorphism classes of $B$-reductions of the $G$-bundle $X_A \times_{X_R} \mathcal{P}$. Consequently, Theorem 1 is equivalent to the assertion that the map $\rho : \text{Fl} \to \text{Spec } R$ admits a section, étale locally on $\text{Spec } R$.

Let $\text{Fl}^\circ$ denote the open subset of $\text{Fl}$ given by the smooth locus of the projection map $\text{Fl} \to \text{Spec } R$. Since every smooth surjection of schemes admits étale-local sections, it will suffice to prove that the projection map $\text{Fl}^\circ \to \text{Spec } R$ is surjective. Note that the special case we have already treated shows that $\rho$ is surjective at the level of $k$-valued points.

Let $y$ be a $k$-valued point of $\text{Spec } R$, and let $\overline{y}$ be a $k$-valued point of $\text{Fl}$ lying over $y$. Let $\mathcal{P}_y$ denote the $G$-bundle on $X$ determined by $y$, so that $\overline{y}$ can be identified with a section $s$ of the projection map $\pi : \mathcal{P}_y/B \to X$. The map $\pi$ is smooth: let $T_\pi$ denote its relative tangent bundle (a vector bundle on $\mathcal{P}_y/B$). Unwinding the definitions, we see that the Zariski tangent space of $\text{Fl} \times_{\text{Spec } R} \{y\}$ at the point $\overline{y}$ can be identified with $H^0(X; s^* T_\pi)$. Using a bit of deformation theory, one can show that the cohomology group $H^1(X; s^* T_\pi)$ controls deformations of the section $s$. In particular, if the group $H^1(X; s^* T_\pi)$ vanishes, then $\overline{y}$ belongs to the smooth locus of $\text{Fl}^\circ$. It will therefore suffice to show that for each $k$-valued point $y$ of $\text{Spec } R$, we can choose a section $s$ such that $H^1(X; s^* T_\pi)$ vanishes. To prove this assertion, we might as well replace $R$ by $k$. We have therefore reduced the proof of Theorem 1 to the following:

**Theorem 5.** Let $\mathcal{P}$ be a $G$-bundle on $X$. Then there exists a section $s$ of the projection map $\pi : \mathcal{P}/B \to X$ such that $H^1(X; s^* T_\pi) \simeq 0$.

We will prove Theorem 5 in the next lecture.

**References**
