Existence of Borel Reductions I (Lecture 14)

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Throughout this lecture, we let k be an algebraically closed field, X an algebraic curve over k, G a smooth affine group scheme over X whose generic fiber G_0 is semisimple and simply connected, B_0 a Borel subgroup of G_0, and B the scheme-theoretic closure of B_0 in G. Our goal is to prove the following version of a theorem of Drinfeld and Simpson:

Theorem 1. *Let R be a finitely generated k-algebra and let $\mathcal{P}$ be a G-bundle on X_R. Then, étale locally on $\operatorname{Spec} R$, the G-bundle $\mathcal{P}$ admits a B-reduction.*

We begin by treating the case where $R = k$. Our starting point is the following:

Lemma 2. *Let $\mathcal{P}$ be a G-bundle on X and let S be a finite set of closed points of X. Then there exists an open set $U \subseteq X$ containing S such that $\mathcal{P}|_U$ is trivial.*

Proof. We first recall that the fraction field $\mathcal{K}_X$ is a field of dimension 1. It follows that any G-bundle on X is automatically trivial at the generic point of X. In particular, if $\mathcal{P}$ is a G-bundle on X, then we can choose a trivialization of $\mathcal{P}$ at the generic point of X. Let us view this trivialization as a map $\eta : \operatorname{Spec} K_X$ fitting into a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K_X & \xrightarrow{\eta} & \mathcal{P} \\ & & \downarrow \\ & & X. \end{array}$$

It follows that η can be extended to a map of X-schemes $U \to \mathcal{P}$, where U is a dense open subset of X, which we can assume is chosen to be as large as possible. We wish to show that after modifying the trivialization η, we can arrange that $S \subseteq U$.

Write $S = \{x_1, \ldots, x_n\}$. Since k is algebraically closed, the G-bundle $\mathcal{P}$ is trivial at the the residue field of each of the points x_i. Since G is smooth, we can extend these trivializations to maps $\eta_i : \operatorname{Spec} \mathcal{O}_{x_i} \to \mathcal{P}$, where $\mathcal{O}_{x_i}$ denotes the complete local ring of the curve X at the point x_i (so that $\mathcal{O}_{x_i}$ is noncanonically isomorphic to a power series ring $k[[t]]$).

For $1 \leq i \leq n$, let K_{x_i} denote the fraction field of $\mathcal{O}_{x_i}$, so that η and η_i determine two different trivializations of $\mathcal{P}|_{\operatorname{Spec} K_{x_i}}$. These trivializations differ by some elements $g_i \in G(K_{x_i})$. Note that g_i belongs to the subgroup $G(\mathcal{O}_{x_i}) \subseteq G(K_{x_i})$ if and only if it is possibly to adjust the trivialization η_i to be compatible with the generic trivialization η: that is, if and only if the point x_i is contained in U.

To complete the proof, we wish to show that we can change the trivialization η to arrange that each g_i belongs to $G(\mathcal{O}_{x_i})$. In other words, we wish to prove that we can choose $g \in G(K_X)$ so that each of the products $gg_i \in G(K_{x_i})$ belongs to $G(\mathcal{O}_{x_i})$.

Each of the fields K_{x_i} admits a topology, with a neighborhood basis of the identity element given by nonzero ideals in the discrete valuation ring $\mathcal{O}_{x_i}$. This determines a topology on each of the groups $G(K_{x_i})$ and therefore also on the product $\prod_{1 \leq i \leq n} G(K_{x_i})$. By construction, the product $\prod_{1 \leq i \leq n} G(\mathcal{O}_{x_i}) g_i^{-1}$ is a nonempty open subset of $\prod_{1 \leq i \leq n} G(K_{x_i})$. It will therefore suffice to prove the following:

$(*)$ The map $G(K_X) \to \prod_{1 \leq i \leq n} G(K_{x_i})$ has dense image.

Note that assertion $(*)$ depends only on the generic fiber G_0 of G. Choose a Borel subgroup B_0^- which is opposite to B_0, so that the intersection $T_0 = B_0 \cap B_0^-$ is a maximal torus in G_0. Let U_0 and U_0^- denote the unipotent radicals of B_0 and B_0^-, respectively. Then the Bruhat decomposition for G implies that the multiplication induces an open immersion

$$U_0^- \times T_0 \times U_0 \hookrightarrow G_0$$

whose image is a Zariski-dense open set $V \subseteq G_0$. We need the following:

Lemma 3. *Let R be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$, let K be the fraction field of R, let Y be a smooth affine K-scheme, and let $U \subseteq Y$ be a dense open set. Then $U(K)$ is dense in $Y(K)$ (where we equip $Y(K)$ with the $\mathfrak{m}$-adic topology).*

It follows from Lemma 3 that each $V(K_{x_i})$ is dense in $G_0(K_{x_i})$. It will therefore suffice to show that the map $V(K_X) \to \prod_{1 \leq i \leq n} V(K_{x_i})$ has dense image. Note that V factors (as a K_X-scheme) as a product of finitely many copies of $\mathbf{G}_a$ and the maximal torus T_0. Moreover, as we saw in the previous lecture, the torus T_0 is induced: that is, it can be written as a product $\prod_{1 \leq j \leq m} \mathrm{Res}^{L_j}\, K_X \mathbf{G}_m$, where $\{L_j\}_{1 \leq j \leq m}$ is a finite collection of separable extensions of K_X. We are therefore reduced to proving that the maps

$$K_X \to \prod_{1 \leq i \leq n} K_{x_i}$$

$$L_j^\times \to \prod_{1 \leq i \leq n} (K_{x_i} \otimes_{K_X} L_j)^\times$$

have dense image, which we leave to the reader. □

Proof of Lemma 3. The assertion is local with respect to the Zariski topology on Y. We may therefore assume without loss of generality that there exists an étale morphism of k-schemes $\phi : Y \to \mathbf{A}^d$, where d is the dimension of Y. Let Z denote the complement of U In Y. Since U is dense, we have $\dim(Z) < d$, so that the image under ϕ of Z is contained in a proper closed subscheme of $\mathbf{A}^d$. We may therefore choose a nonzero polynomial $f(x_1, \ldots, x_d)$ which vanishes on the points of $\phi(Z(K))$, so that $\phi(Z(K))$ cannot contain any nonempty open subset of K^n. It follows from Hensel's lemma that ϕ induces an open map $Y(K) \to K^d$, so that $Z(K)$ cannot contain any open subset of $Y(K)$ and therefore $U(K)$ is dense in $Y(K)$, as desired. □

We now return to the proof of Theorem 1 in the special case where $R = k$. Let $\mathcal{P}$ be a G-bundle on X. Then $\mathcal{P}$ is equipped with a free action of G (in the category of X-schemes), and in particular a free action of B. We let $\mathcal{P}/B$ denote the quotient of $\mathcal{P}$ by the action of B. We have a canonical map $\pi : \mathcal{P}/B \to X$, and we can identify B-reductions of $\mathcal{P}$ with sections of the map π.

Remark 4. By the quotient $\mathcal{P}/B$, we refer to the quotient of $\mathcal{P}$ by B in the category of fppf sheaves on X. It follows from a general theorem of Artin that such a quotient is always representable by an algebraic space ([**?**]). In fact, one can show that $\mathcal{P}/B$ is representable by a scheme, but this is not really important.

The generic fiber of π can be identified with the quotient G_0/B_0 in the category of K_X-schemes. Since B_0 was defined to be a Borel subgroup of G_0, the quotient G_0/B_0 is *proper* over $\mathrm{Spec}\, K_X$. It follows that the map $\pi : \mathcal{P}/B \to X$ is *generically proper*: that is, it induces a proper map $(\mathcal{P}/B) \times_X W \to W$ for some dense open subset $W \subseteq X$. The complement $X - W$ consists of finitely many closed points $x_1, \ldots, x_n \in X$. Applying Lemma 2, we can choose an open subset $U \subseteq X$ containing each x_i such that $\mathcal{P}|_U$ is trivial. In particular, the G-bundle $\mathcal{P}|_U$ admits a reduction to B, classified by a map $s : U \to \mathcal{P}/B$ which is a section of π. Since every point of $X - U$ belongs to W, the map s extends uniquely to a map $\overline{s} : X \to \mathcal{P}/B$ using the valuative criterion for properness, which we can identify with a B-reduction of $\mathcal{P}$. This completes the proof of Theorem 1 in the special case $R = k$.

Let us now turn to the general case. Let R be a finitely generated k-algebra and let $\mathcal{P}$ be a G-bundle on X_R. As before, we let $\mathcal{P}/B$ denote the quotient of $\mathcal{P}$ by the action of B. Let Fl be the R-scheme obtained by Weil restriction of $\mathcal{P}/B$ along the projection map $X_R \to \operatorname{Spec} R$ (see [2] for a careful discussion). In other words, Fl is the R-scheme whose set of A-valued points $\mathrm{Fl}(A)$ can be identified with the set of commutative diagrams

$$\begin{array}{ccc} X_A & \longrightarrow & \mathcal{P}/B \\ & \searrow & \downarrow \\ & & X_R. \end{array}$$

Unwinding the definitions, we see that there is a bijective correspondence between $\mathrm{Fl}(A)$ and the set of isomorphism classes of B-reductions of the G-bundle $X_A \times_{X_R} \mathcal{P}$. Consequently, Theorem 1 is equivalent to the assertion that the map $\rho : \mathrm{Fl} \to \operatorname{Spec} R$ admits a section, étale locally on $\operatorname{Spec} R$.

Let Fl° denote the open subset of Fl given by the smooth locus of the projection map $\mathrm{Fl} \to \operatorname{Spec} R$. Since every smooth surjection of schemes admits étale-local sections, it will suffice to prove that the projection map $\mathrm{Fl}^\circ \to \operatorname{Spec} R$ is surjective. Note that the special case we have already treated shows that ρ is surjective at the level of k-valued points.

Let y be a k-valued point of $\operatorname{Spec} R$, and let $\overline{y}$ be a k-valued point of Fl lying over y. Let $\mathcal{P}_y$ denote the G-bundle on X determined by y, so that $\overline{y}$ can be identified with a section s of the projection map $\pi : \mathcal{P}_y/B \to X$. The map π is smooth: let T_π denote its relative tangent bundle (a vector bundle on $\mathcal{P}_y/B$. Unwinding the definitions, we see that the Zariski tangent space of $\mathrm{Fl} \times_{\operatorname{Spec} R} \{y\}$ at the point $\overline{y}$ can be identified with $\mathrm{H}^0(X; s^* T_\pi)$. Using a bit of deformation theory, one can show that the cohomology group $\mathrm{H}^1(X; s^* T_\pi)$ controls deformations of the section s. In particular, if the group $\mathrm{H}^1(X; s^* T_\pi)$ vanishes, then $\overline{y}$ belongs to the smooth locus of Fl°. It will therefore suffice to show that for each k-valued point y of $\operatorname{Spec} R$, we can choose a section s such that $\mathrm{H}^1(X; s^* T_\pi)$ vanishes. To prove this assertion, we might as well replace R by k. We have therefore reduced the proof of Theorem 1 to the following:

Theorem 5. *Let $\mathcal{P}$ be a G-bundle on X. Then there exists a section s of the projection map $\pi : \mathcal{P}/B \to X$ such that* $\mathrm{H}^1(X; s^* T_\pi) \simeq 0$.

We will prove Theorem 5 in the next lecture.

References

[1] Drinfeld, V. and C. Simpson. *B-Structures on G-bundles and Local Triviality.*

[2] Nitsure, N. *Construction of Hilbert and Quot schemes.* Fundamental algebraic geometry: Grothendieck's FGA explained, Mathematical Surveys and Monographs 123, American Mathematical Society 2005, 105-137.