Existence of Generic Trivializations (Lecture 13)

March 5, 2014

Let $k$ be an algebraically closed field, $X$ an algebraic curve over $k$, and $G$ a smooth affine group scheme over $X$ whose generic fiber is semisimple and simply connected. Recall that our goal is to prove (under some hypotheses on the generic fiber of $G$) that the map

$$\text{Ran}_G(X) \to \text{Bun}_G(X)$$

is a universal homology equivalence. For this, it will suffice to show that the following assertion holds for every $R \in \text{Ring}_k$ and every $G$-bundle $P$ on $X_R$:

(**) The projection map $\text{Sect}(P) = \text{Spec} R \times_{\text{Bun}_G(X)} \text{Ran}_G(X) \to \text{Spec} R$ is a universal homology equivalence.

We begin with two observations:

(a) If $P$ and $P'$ are $G$-bundles on $X_R$ which are isomorphic over an open set $X_R - |S|$ for some finite subset $S \subseteq X(R)$, then $P$ satisfies (**)) if and only if $P'$ satisfies (**). This is the upshot of the previous lecture.

(b) The assertion that $P$ satisfies (**)) can be tested locally with respect to the fppf topology on $\text{Spec} R$ (this follows from elementary formal properties of $\ell$-adic cohomology).

To make use of (a), it is convenient to introduce the following:

**Definition 1.** Let $R \in \text{Ring}_k$ and let $U \subseteq X_R$ be an open set. We will say that $U$ is full if the projection map $U \to \text{Spec} R$ is surjective (that is, $U$ intersects each fiber of the map $X_R \to \text{Spec} R$).

We will say that a $G$-bundle on $P_R$ is generically trivial if it is trivial over some full open set $U \subseteq X_R$.

To deduce (**)) from (a) and (b), it will suffice to prove the following:

(i) For every $G$-bundle $P$ on $X_R$, there is a faithfully flat ring étale map $\text{Spec} R' \to \text{Spec} R$ such that $P \times_{X_R} X_{R'}$ is generically trivial.

(ii) Let $P$ be a generically trivial $G$-bundle on $X_R$. Then there exists a faithfully flat map $\text{Spec} R' \to \text{Spec} R$ and a finite subset $S \subseteq X(R')$ such that $P$ is trivial when restricted to $X_{R'} - |S|$.

(iii) Every trivial $G$-bundle $P$ satisfies (**))

In this lecture, we will prove (ii) and begin the proof of (i). Assertion (ii) is an immediate consequence of the following:

**Proposition 2.** Let $R$ be a finitely generated $k$-algebra and let $U \subseteq X_R$ be a full open subset. Then there exists a faithfully flat map $R \to R'$ and a finite subset $S \subseteq X(R')$ such that $X_{R'} - |S|$ is contained in the inverse image $U \times_{X_R} X_{R'}$. 


Proof. Choose a $k$-valued point $y$ of $\text{Spec } R$. Since $U$ is full, we can choose a $k$-valued point $(x, y)$ of $U \subseteq X \times_{\text{Spec } k} \text{Spec } R$ lying over $y$. Let $U(x)$ denote the intersection $(\{x\} \times_{\text{Spec } k} \text{Spec } R) \cap U$. Then the projection map $U(x) \to \text{Spec } R$ is an open immersion whose image contains the point $y$. Since the desired assertion is Zariski local on $\text{Spec } R$, we may assume that without loss of generality that $U(x) \to \text{Spec } R$ is surjective: that is, that $U$ contains the product $\{x\} \times \text{Spec } R$.

Choose a closed subscheme $K \subseteq X_R$ whose underlying topological space is the complement of $U$ (for example, we could endow $K$ with the reduced structure). Let $J_K \subseteq \mathcal{O}_{X_R}$ denote the ideal sheaf of $K$. For each integer $n \geq 0$, let $J_K(nx)$ denote the tensor product of $J_K$ with the pullback of the ample invertible sheaf $\mathcal{O}_X(nx)$ on $X$. For $n \gg 0$, the sheaf $J_K(nx)$ is generated by global sections. Choose a point $x' \in X(k)$ such that $(x', y)$ is contained in $U$ and a global section $f$ of $J_K(nx)$ which does not vanish at $(x', y)$. Without loss of generality (passing to a Zariski open neighborhood of $y$ if necessary) we may assume that $f$ does not vanish identically on any fiber of the projection map $X_R \to \text{Spec } R$. Let $\mathcal{I}$ identify $f$ with a section of the line bundle $\mathcal{O}_{X_R}(nx)$, so that the vanishing locus of $f$ is a closed subscheme $D \subseteq X_R$. By construction, this closed subscheme contains $K$ and the projection map $D \to \text{Spec } R$ is finite and flat of degree $n$. Replacing $U$ by $X_R - D$, we are reduced to proving Proposition 2 under the following additional assumption:

\[ (*) \text{ There exists a closed subscheme } D \subseteq X_R \text{ such that } D \to \text{Spec } R \text{ is finite flat of degree } n \geq 0, \text{ and } U = X - D. \]

We now proceed by induction on $n$. If $n = 0$, then $U = X_R$ and there is nothing to prove. Otherwise, the map $D \to \text{Spec } R$ is finite flat. Replacing $\text{Spec } R$ by $D$, we can reduce to the case where the map $D \to \text{Spec } R$ admits a section $s$. Let $D_0 \subseteq X_R$ be the image of the section $s$. Then $D_0$ is contained in $D$, so we can write $D$ as a divisorial sum $D_0 + D_1$ where $D_1 \subseteq X_R$ has degree $(n-1)$ over $R$. Using the inductive hypothesis, we can assume that there exists a finite set $S_0 \subseteq X(R)$ such that $X_R - |S_0| \subseteq X - D_1$. We now complete the proof by taking $S = S_0 \cup \{s\}$. \qed

We now turn to the proof of (i). In the case where the group scheme $G$ is split, a strong version of this result was proven by Drinfeld and Simpson in [?]. Let us briefly describe the strategy of proof. Fix a closed point $u \in \text{Spec } R$, and let $\mathcal{P}_u$ denote the fiber of $\mathcal{P}$ at $u$ (so that $\mathcal{P}_u$ is a $G$-bundle on $X$). Let $\mathcal{K}_X$ denote the fraction field of $X$. Since $\mathcal{K}_X$ has dimension 1 (see [?]), any $G$-bundle on $\text{Spec } \mathcal{K}_X$ is automatically trivial. It follows that $\mathcal{P}_u$ is trivial at the generic point of $X$, and therefore admits a trivialization $\eta_u$ on a dense open subset $U \subseteq X$.

One would now like to show that it is possible to “deform” the trivialization $\eta_u$ to obtain trivializations of $\mathcal{P}_v$ for points $v \in \text{Spec } R$ which are sufficiently “near” to $u$. However, there is an obstacle: the trivialization $\eta_u$ is defined only on a dense open subset $U \subseteq X$. The collection of trivializations is therefore a torsor for the group $G(U)$, which is an unwieldy infinite-dimensional object which is ill-suited for study by standard tools of deformation theory. On the other hand, there is no guarantee that $\eta_u$ can be extended to a trivialization of $\mathcal{P}_u$ over the entire curve $X$ (since $G$-bundles on $X$ can certainly be globally nontrivial).

To circumvent this difficulty, Drinfeld and Simpson first sought after a weaker structure on $\mathcal{P}_u$: namely, a reduction of structure group from $G$ to a Borel subgroup $B \subseteq G$. Any trivialization of $\mathcal{P}_u$ over a dense open subset $U \subseteq X$ determines a $B$-structure on $\mathcal{P}_u|_U$, which can then be extended to a $B$-structure on the entire torsor $\mathcal{P}_u$ using the valuative criterion for properness (since the quotient $G/B$ is proper, at least in the case where $G$ has good reduction everywhere). One might then hope to use deformation theory to show that a $B$-structure on $\mathcal{P}_u$ can be extended to a $B$-structure on $\mathcal{P}$ in a neighborhood of $u$ (at least if the original $B$-structure is well-chosen). On the other hand, $B$-structures are easy to analyze, since $B$ is a solvable algebraic group.

Our proof will follow the same basic strategy. However, we must be careful about the meaning of “Borel subgroup” if the group scheme $G$ is not assumed to be split.

Notation 3. Let $G_0$ denote the generic fiber of $G$. Then $G_0$ is a reductive algebraic group over the fraction field $K_X$, which has dimension 1. It follows that $G_0$ is quasi-split: that is, we can choose a Borel subgroup $B_0 \subseteq G_0$ which is defined over $K_X$. We let $B$ denote the scheme-theoretic closure of $B_0$ in $G$. 

2
Warning 4. The scheme $B$ is flat over $X$ and is closed under multiplication and inversion in $G$: in particular, it can be regarded as a flat affine group scheme over $X$. However, there is no reason to expect that $B$ should be smooth over $X$, or that the fibers of $B$ should be connected.

Exercise 5. Let $x \in X$ be a closed point for which the fiber $G_x$ is reductive. Show that $B_x$ is a Borel subgroup of $G_x$.

Now suppose we are given a finitely generated $k$-algebra $R$ and a $G$-bundle $\mathcal{P}$ over $X_R$. By definition, a $B$-reduction of $\mathcal{P}$ is a pair $(\mathfrak{Q}, \alpha)$, where $\mathfrak{Q}$ is a $B$-bundle on the relative curve $X_R$, and $\alpha$ is an isomorphism of $\mathcal{P}$ with the induced $G$-bundle $(\mathfrak{Q} \times_X G)/B$.

Warning 6. If $B$ is not smooth over $X$, we should take care in specifying what we mean by a $B$-bundle. In this context, we mean an $X_R$-scheme $\mathfrak{Q}$ equipped with an action of $B$ for which there exists an fpqc surjection $U \to X_R$ and a $B$-equivariant isomorphism $U \times_{X_R} \mathfrak{Q} \simeq U \times X B$ (in general, there is no reason to expect that such a trivialization can be found étale-locally).

The proof of (i) can be broken into two steps:

(i') For every $G$-bundle $\mathcal{P}$ on $X_R$, there exists an étale surjection $\text{Spec} R' \to \text{Spec} R$ such that $\mathcal{P} \times_{X_R} X_R'$ admits a $B$-reduction.

(i'') Every $B$-bundle on $X_R$ is generically trivial.

We will prove (i'') in this lecture, and postpone (i') until the next.

Let $\text{rad}_u(B_0)$ denote the unipotent radical of $B_0$ and let $T_0 = B_0/\text{rad}_u(B_0)$ denote the quotient torus. Then the group scheme $B_0$ fits into an exact sequence

$$0 \to \text{rad}_u(B_0) \to B_0 \to T_0 \to 0.$$  

Fix an algebraic closure $\overline{K}_X$ of $K_X$ and let $\Lambda$ denote collection of all maps from $T_0$ into the multiplicative group $G_m$ which are defined over $\overline{K}_X$. Then $\Lambda$ is a finite free abelian group with an action of $\text{Gal}(\overline{K}_X/K_X)$. Since $G_0$ is simply connected, the lattice $\Lambda$ has a canonical basis given by the simple weights of $G_0$ (which are in bijection with the vertices of the Dynkin diagram of $G_0$); this basis is stable under the action of $\text{Gal}(\overline{K}_X/K_X)$. Let $I$ denote the set of orbits for $\text{Gal}(\overline{K}_X/K_X)$. For each $I \in \mathcal{I}$, let $\Gamma_I \subseteq \text{Gal}(\overline{K}_X/K_X)$ be the (open) stabilizer of some element of $I$ and let $K_I \subseteq \overline{K}_X$ be the fixed field of $\Gamma_I$, so that $K_I$ is a finite separable extension of $K_X$ of degree equal to the cardinality of $I$. Unwinding the definition, we see that the torus $T_0$ is given by

$$\text{Hom}(\Lambda, G_m) = \text{Hom}(\bigoplus_{I \in \mathcal{I}} \mathbb{Z}[I], G_m) = \prod_{I \in \mathcal{I}} \text{Res}^K_{K_X} G_m.$$  

Here $\text{Res}^K_{K_X} G_m$ denotes the Weil restriction of the multiplicative group $G_m$ along the map $\text{Spec} K_I \to \text{Spec} K_X$.

For each $I \in \mathcal{I}$, we can identify $K_I$ with the fraction field of an algebraic curve $X_I$ equipped with a generically étale map $\pi_I : X_I \to X$. By a direct limit argument, we can choose a dense open set $U \subseteq X$ with the following properties:

1. Each of the maps $U_I = X_I \times_X U \to U$ is étale.
2. The group scheme $B_U = B \times_X U$ fits into an exact sequence

$$0 \to N \to B_U \to T_U \to 0$$

where $T_U \simeq \prod_{I \in \mathcal{I}} \text{Res}^U_{U_I} G_m$.
3. The group scheme $N$ admits a finite filtration by copies of the additive group $G_a$.  

3
Let $\mathcal{Q}$ be a $B$-bundle on $X_R$. Then $\mathcal{Q}$ determines a $T_U$-bundle over the open set $U_R \subseteq X_R$, which we can identify with a finite collection of line bundles $\mathcal{L}_I$ over the schemes $U_{IR} = U_I \times_{\text{Spec} k} \text{Spec} R$. Each of these line bundles can be trivialized locally with respect to the Zariski topology. Working Zariski-locally on $\text{Spec} R$, we may assume that each $\mathcal{L}_I$ is trivial over a full open set $V_I \subseteq U_{IR}$. Then $W = U_R - \bigcup_{I \in J} \text{Im}(U_{IR} - V_I \to U_R)$ is a full open subset of $X_R$. Working locally on $\text{Spec} R$, we may assume that $W$ contains a full affine open subset $W_0 \subseteq W$. By construction, the restriction $\mathcal{Q}|_{W_0}$ determines a trivial $T_U$-bundle on $W_0$. It follows that the structure group of $\mathcal{Q}|_{W_0}$ can be reduced to the subgroup $N \subseteq B_U$. Using assertion (3) and our assumption that $W_0$ is affine, we see that every $N$-bundle on $W_0$ is trivial. It follows that $\mathcal{Q}|_{W_0}$ is trivial, so that $\mathcal{Q}$ is generically trivial as desired.