

Higher Category Theory (Lecture 5)

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In the previous lecture, we outlined some approaches to describing the cohomology of the classifying space of G -bundles $\text{Bun}_G(X)$ on a Riemann surface X . For example, we asserted that the cochain complex $C^*(\text{Bun}_G(X); \mathbf{Q})$ is quasi-isomorphic to a continuous tensor product $\otimes_{x \in X} C^*(BG_x; \mathbf{Q})$. Here it is vital that we work at the level of cochains, rather than cohomology: there is no corresponding procedure to recover the cohomology ring $H^*(\mathcal{M}; \mathbf{Q})$ from the graded rings $H^*(BG_x; \mathbf{Q})$. Consequently, even if our ultimate interest is in understanding the cohomology ring $H^*(\mathcal{M}; \mathbf{Q})$, it will be helpful to have a good way of thinking about chain-level constructions in homological algebra.

Let Λ be a commutative ring. Throughout this lecture, we let $\text{Chain}(\Lambda)$ denote the abelian category whose objects are chain complexes

$$\cdots \rightarrow V_2 \rightarrow V_1 \rightarrow V_0 \rightarrow V_{-1} \rightarrow V_{-2} \rightarrow \cdots$$

of Λ -modules. We will always employ *homological* conventions when discussing chain complexes (so that differential on a chain complex lowers degree). If V_* is a chain complex, then its *homology* $H_*(V_*)$ is given by

$$H_n(V_*) = \{x \in V_n : dx = 0\} / \{x \in V_n : (\exists y \in V_{n-1})[x = dy]\}.$$

Any map of chain complexes $\alpha : V_* \rightarrow W_*$ induces a map $H_*(V_*) \rightarrow H_*(W_*)$. We say that α is a *quasi-isomorphism* if it induces an isomorphism on homology.

For many purposes, it is convenient to treat quasi-isomorphisms as if they are isomorphisms (emphasizing the idea that a chain complex is just a vessel for carrying information about its homology). One can make this idea explicit using Verdier's theory of *derived categories*. The derived category $\mathcal{D}(\Lambda)$ can be described as the category obtained from $\text{Chain}(\Lambda)$ by formally inverting all quasi-isomorphisms.

The theory of derived categories is a very useful tool in homological algebra, but has a number of limitations. Many of these stem from the fact that $\mathcal{D}(\Lambda)$ is not very well-behaved from a categorical point of view. The category $\mathcal{D}(\Lambda)$ does not generally have limits or colimits, even of very simple types. For example, a morphism $f : X \rightarrow Y$ in $\mathcal{D}(\Lambda)$ generally does not have a cokernel in $\mathcal{D}(\Lambda)$. However, there is a substitute: every morphism f in $\mathcal{D}(\Lambda)$ fits into a "distinguished triangle"

$$X \xrightarrow{f} Y \rightarrow \text{Cn}(f) \rightarrow \Sigma X.$$

Here we refer to $\text{Cn}(f)$ is called the *cone* of f , and it behaves in some respects like a cokernel: every map $g : Y \rightarrow Z$ such that $g \circ f = 0$ factors through $\text{Cn}(f)$, though the factorization is generally not unique. The object $\text{Cn}(f) \in \mathcal{D}(\Lambda)$ (and, in fact, the entire diagram above) is well-defined up to isomorphism, but *not* up to canonical isomorphism: there is no functorial procedure for constructing the cone $\text{Cn}(f)$ from the data of a morphism f in the category \mathcal{D} . And this is only a very simple example: for other types of limits and colimits (such as taking invariants or coinvariants with respect to the action of a group), the situation is even worse.

Let $f, g : V_* \rightarrow W_*$ be maps of chain complexes. Recall that a *chain homotopy* from f_* to g_* is a collection of maps $h_n : V_n \rightarrow W_{n+1}$ such that $f_n - g_n = d \circ h_n + h_{n-1} \circ d$. In this case, we say that f and g are *chain-homotopic*. Chain-homotopic maps induce the same map from $H_*(V_*)$ to $H_*(W_*)$, and have the same

image in the derived category $\mathcal{D}(\Lambda)$. In fact, there is an alternative description of the derived category $\mathcal{D}(\Lambda)$, which places an emphasis on the notion of chain-homotopy rather than quasi-isomorphism. More precisely, one can define a category $\mathcal{D}'(\Lambda)$ equivalent to $\mathcal{D}(\Lambda)$ as follows:

Definition 1. • The objects of $\mathcal{D}'(\Lambda)$ are the K -injective chain complexes of Λ -modules, in the sense of [5]. A chain complex V_* is K -injective if, for every chain complex $W_* \in \text{Chain}(\Lambda)$ and every subcomplex $W'_* \subseteq W_*$ which is quasi-isomorphic to W_* , every chain map $f : W'_* \rightarrow V_*$ can be extended to a chain map $\bar{f} : W_* \rightarrow V_*$.

- A morphism from V_* to W_* in $\mathcal{D}'(\Lambda)$ is a chain-homotopy equivalence class of chain maps from V_* to W_* .

Remark 2. If $V_* \in \text{Chain}(\Lambda)$ is K -injective, then each V_n is an injective Λ -module. The converse holds if $V_n \simeq 0$ for $n \gg 0$ or if the commutative ring Λ has finite injective dimension (for example, if $\Lambda = \mathbf{Z}$), but not in general. For example, the chain complex of $\mathbf{Z}/4\mathbf{Z}$ -modules

$$\cdots \rightarrow \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \rightarrow \cdots$$

is not K -injective.

From the perspective of Definition 1, categorical issues with the derived category stem from the fact that we are identifying chain-homotopic morphisms in $\mathcal{D}'(\Lambda)$ without remembering *how* they are chain-homotopic. For example, suppose that we wish to construct the cone of a morphism $[f] : V_* \rightarrow W_*$ in $\mathcal{D}'(\Lambda)$. By definition, $[f]$ is an equivalence class of chain maps from V_* to W_* . If we choose a representative f for the equivalence class $[f]$, then we can construct the mapping cone $\text{Cn}(f)$ by equipping the direct sum $W_* \oplus V_{*-1}$ with a differential which depends on f . If h is a chain-homotopy from f to g , we can use h to construct an isomorphism of chain complexes $\alpha_h : \text{Cn}(f) \simeq \text{Cn}(g)$. However, the isomorphism α_h depends on h : different choices of chain homotopy can lead to different isomorphisms, even up to chain-homotopy.

It is possible to correct many of the deficiencies of the derived category by keeping track of more information. To do so, it is useful to work with mathematical structures which are a bit more elaborate than categories, where the primitive notions include not only “object” and “morphism” but also a notion of “homotopy between morphisms.” Before giving a general definition, let us spell out the structure that is visible in the example of chain complexes over Λ .

Construction 3. We define a sequence of sets S_0, S_1, S_2, \dots as follows:

- Let S_0 denote the set of *objects* under consideration: in our case, these are chain complexes X of K -injective chain complexes of Λ -modules (strictly speaking this is not a set but a proper class, because we are trying to describe a “large” category).
- Let S_1 denote the set of *morphisms* under consideration. That is, S_1 is the collection of all chain maps $f : X \rightarrow Y$, where X and Y are chain complexes of injective abelian groups.
- Let S_2 denote the set of all pairs consisting of a non-necessarily commutative diagram

$$\begin{array}{ccc} & Y & \\ f_{01} \nearrow & & \searrow f_{12} \\ X & \xrightarrow{f_{02}} & Z \end{array}$$

together with a chain homotopy f_{012} from f_{02} to $f_{12} \circ f_{01}$. Here X , Y , and Z are chain complexes of injective abelian groups.

- More generally, we let S_n denote the collection of all n -tuples $\{X(0), X(1), \dots, X(n)\}$ of chain complexes of injective abelian groups, together with chain maps $f_{i,j} : X(i) \rightarrow X(j)$ which are compatible with composition *up to coherent homotopy*. More precisely, this means that for every subset $I = \{i_- < i_1 < \dots < i_m < i_+\} \subseteq \{0, \dots, n\}$, we supply a collection of maps $f_I : X(i_-)_k \rightarrow X(i_+)_{k+m}$ satisfying the identities

$$d(f_I(x)) = (-1)^m f_I(dx) + \sum_{1 \leq j \leq m} (-1)^j (f_{I - \{i_j\}}(x) - (f_{\{i_j, \dots, i_+\}} \circ f_{\{i_-, \dots, i_j\}})(x)).$$

Suppose we are given an element $(\{X(i)\}_{0 \leq i \leq n}, \{f_I\})$ of S_n . Then for $0 \leq i \leq n$, we can regard $X(i)$ as an element of S_0 . If we are given a pair of integers $0 \leq i < j \leq n$, then $f_{\{i,j\}}$ is a chain map from $X(i)$ to $X(j)$, which we can regard as an element of S_1 . More generally, given any nondecreasing map $\alpha : \{0, \dots, m\} \rightarrow \{0, \dots, n\}$, we can define a map $\alpha^* : S_n \rightarrow S_m$ by the formula

$$\alpha^*(\{X(j)\}_{0 \leq j \leq n}, \{f_I\}) = (\{X(\alpha(j))\}_{0 \leq j \leq m}, \{g_J\}),$$

where

$$g_J(x) = \begin{cases} f_{\alpha(J)}(x) & \text{if } \alpha|_J \text{ is injective} \\ x & \text{if } J = \{j, j'\} \text{ and } \alpha(j) = \alpha(j') \\ 0 & \text{otherwise.} \end{cases}$$

This motivates the following:

Definition 4. A *simplicial set* X_\bullet consists of the following data:

- For every integer $n \geq 0$, a set X_n (called the *set of n -simplices of X_\bullet*).
- For every nondecreasing map of finite sets $\alpha : \{0, 1, \dots, m\} \rightarrow \{0, 1, \dots, n\}$, a map of sets $\alpha^* : X_n \rightarrow X_m$.

This data is required to be compatible with composition: that is, we have

$$\text{id}^*(x) = x \quad (\alpha \circ \beta)^*(x) = \beta^*(\alpha^*(x))$$

whenever α and β are composable nondecreasing maps.

If X_\bullet is a simplicial set, we will refer to X_n as the *set of n -simplices of X_\bullet* .

Example 5 (The Nerve of a Category). Let \mathcal{C} be a category. We can associate to \mathcal{C} a simplicial set $N(\mathcal{C})_\bullet$, whose n -simplices are given by chains of composable morphisms

$$C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_n$$

in \mathcal{C} . We refer to $N(\mathcal{C})_\bullet$ as the *nerve* of the category \mathcal{C} .

Example 6. Let Λ be a commutative ring and let $\text{Chain}'(\Lambda)$ denotes the full subcategory of $\text{Chain}(\Lambda)$ spanned by the K -injective chain complexes of Λ -modules. Construction 3 yields a simplicial set $\{S_n\}_{n \geq 0}$ which we will denote by Mod_Λ . The simplicial set Mod_Λ can be regarded as an enlargement of the nerve $N(\text{Chain}'(\Lambda))_\bullet$ (more precisely, we can identify $N(\text{Chain}'(\Lambda))_\bullet$ with the simplicial subset of Mod_Λ whose n -simplices are pairs $(\{X(i)\}_{0 \leq i \leq n}, \{f_I\})$ for which $f_I = 0$ whenever I has cardinality > 2).

The construction $\text{Chain}'(\Lambda) \mapsto \text{Mod}_\Lambda$ can be regarded as a variant of Example 5 which takes into account the structure of $\text{Chain}'(\Lambda)$ as a *differential graded* category. We refer to [3] for more details.

From the nerve of a category \mathcal{C} , we can recover \mathcal{C} up to isomorphism. For example, the objects of \mathcal{C} are just the 0-simplices of $N(\mathcal{C})_\bullet$ and the morphisms of \mathcal{C} are just the 1-simplices of $N(\mathcal{C})_\bullet$. Moreover, given a

pair of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} , the composition $h = g \circ f$ is the unique 1-morphism in \mathcal{C} for which there exists a 2-simplex $\sigma \in N(\mathcal{C})_2$ satisfying

$$\alpha_0^*(\sigma) = g \quad \alpha_1^*(\sigma) = h \quad \alpha_2^*(\sigma) = f,$$

where $\alpha_i : \{0, 1\} \rightarrow \{0, 1, 2\}$ denotes the unique injective map whose image does not contain i .

If \mathcal{C} and \mathcal{D} are categories, then there is a bijective correspondence between functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and maps of simplicial sets $N(\mathcal{C})_\bullet \rightarrow N(\mathcal{D})_\bullet$. We can summarize the situation as follows: the construction $\mathcal{C} \mapsto N(\mathcal{C})_\bullet$ furnishes a fully faithful embedding from the category of (small) categories to the category of simplicial sets. It is therefore natural to ask about the essential image of this construction: which simplicial sets arise as the nerves of categories? To answer this question, we need a bit of terminology:

Notation 7. Let X_\bullet be a simplicial set. For $0 \leq i \leq n$, we define a set $\Lambda_i^n(X_\bullet)$ as follows:

- To give an element of $\Lambda_i^n(X_\bullet)$, one must give an element $\sigma_J \in X_m$ for every subset $J = \{j_0 < \dots < j_m\} \subseteq \{0, \dots, n\}$ which does not contain $\{0, 1, \dots, i-1, i+1, \dots, n\}$. These elements are subject to the compatibility condition $\sigma_I = \alpha^* \sigma_J$ whenever $I = \{i_0 < \dots < i_\ell\} \subseteq \{j_0 < \dots < j_m\}$ and α satisfies $i_k = j_{\alpha(k)}$.

More informally, $\Lambda_i^n(X_\bullet)$ is the set of “partially defined” n -simplices of X_\bullet , which are missing their interior and a single face. There is an evident restriction map $X_n \rightarrow \Lambda_i^n(X_\bullet)$.

Proposition 8. *Let X_\bullet be a simplicial set. Then X_\bullet is isomorphic to the nerve of a category if and only if, for each $0 < i < n$, the restriction map $X_n \rightarrow \Lambda_i^n(X_\bullet)$ is bijective.*

For example, the bijectivity of the map $X_2 \rightarrow \Lambda_1^2(X_\bullet)$ encodes the existence and uniqueness of composition: it says that every pair of composable morphisms $f : C \rightarrow D$ and $g : D \rightarrow E$ can be completed uniquely to a commutative diagram

$$\begin{array}{ccc} & D & \\ f \nearrow & & \searrow g \\ C & \overset{h}{\dashrightarrow} & E. \end{array}$$

Example 9. Let Z be a topological space. We can associate to Z a simplicial set $\text{Sing}(Z)_\bullet$, whose n -simplices are continuous maps $\Delta^n \rightarrow Z$ (here Δ^n denotes the standard n -simplex: that is, the convex hull of the standard basis for \mathbb{R}^{n+1}). The simplicial set $\text{Sing}(Z)_\bullet$ is called the *singular simplicial set* of Z .

From the perspective of homotopy theory, the singular simplicial set $\text{Sing}(Z)_\bullet$ is a complete invariant of X . More precisely, from $\text{Sing}(Z)_\bullet$ one can functorially construct a topological space which is (weakly) homotopy equivalent to Z . Consequently, the simplicial set $\text{Sing}(Z)_\bullet$ can often serve as a surrogate for Z . For example, there is a combinatorial recipe for extracting the homotopy groups of Z directly from $\text{Sing}(Z)_\bullet$. However, this recipe works only for a special class of simplicial sets:

Definition 10. Let X_\bullet be a simplicial set. We say that X is a *Kan complex* if, for $0 \leq i \leq n$, the map $X_n \rightarrow \Lambda_i^n(X_\bullet)$ is surjective.

Example 11. For any topological space Z , the singular simplicial set $\text{Sing}(Z)_\bullet$ is a Kan complex. To see this, let H denote the topological space obtained from the standard n -simplex Δ^n by removing the interior and the i th face. Then $\Lambda_i^n(\text{Sing}(Z)_\bullet)$ can be identified with the set of continuous maps from H into Z . Any continuous map from H into Z can be extended to a map from Δ^n into Z , since H is a retract of Δ^n .

The converse of Example 11 fails: not every Kan complex is isomorphic to the singular simplicial set of a topological space. However, every Kan complex X_\bullet is *homotopy equivalent* to the singular simplicial set of a topological space, which can be constructed explicitly from X_\bullet . In fact, something stronger is true: the construction $Z \mapsto \text{Sing}(Z)_\bullet$ induces an equivalence from the homotopy category of nice spaces (say, CW complexes) to the homotopy category of Kan complexes (which can be defined in a purely combinatorial way).

Example 12. A *simplicial Λ -module* is a simplicial set X_\bullet for which each of the sets X_n is equipped with the structure of a Λ -module, and each of the maps $\alpha^* : X_n \rightarrow X_m$ is a Λ -module homomorphism. One can show that every simplicial Λ -module is a Kan complex, so that one has homotopy groups $\{\pi_n X_\bullet\}_{n \geq 0}$. According to the classical *Dold-Kan correspondence*, the category of simplicial Λ -modules is equivalent to the category $\text{Chain}_{\geq 0}(\Lambda) \subseteq \text{Chain}(\Lambda)$ of nonnegatively graded chain complexes of Λ -modules. Under this equivalence, the homotopy groups of a simplicial Λ -module X_\bullet can be identified with the homology groups of the corresponding chain complex.

The hypothesis of Proposition 8 resembles the definition of a Kan complex, but is different in two important respects. Definition 10 requires that every element of $\Lambda_i^n(X_\bullet)$ can be extended to an n -simplex of X . Proposition 8 requires this condition only in the case $0 < i < n$, but demands that the extension be unique. Neither condition implies the other, but they admit a common generalization:

Definition 13. A simplicial set X_\bullet is an ∞ -category if, for each $0 < i < n$, the map $X_n \rightarrow \Lambda_i^n(X_\bullet)$ is surjective.

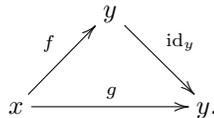
Remark 14. A simplicial set X_\bullet satisfying the requirement of Definition 13 is also referred to as a *quasi-category* or a *weak Kan complex* in the literature.

Example 15. Any Kan complex is an ∞ -category. In particular, for any topological space Z , the singular simplicial set $\text{Sing}(Z)_\bullet$ is an ∞ -category.

Example 16. For any category \mathcal{C} , the nerve $N(\mathcal{C})_\bullet$ is an ∞ -category.

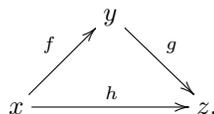
By virtue of the discussion following Example 5, no information is lost by identifying a category \mathcal{C} with the simplicial set $N(\mathcal{C})_\bullet$. It is often convenient to abuse notation by identifying \mathcal{C} with its nerve, thereby viewing a category as a special type of ∞ -category. We will generally use category-theoretic notation and terminology when discussing ∞ -categories. Here is a brief sampler; for a more detailed discussion of how the basic notions of category theory can be generalized to this setting, we refer the reader to the first chapter of [2].

- Let $\mathcal{C} = \mathcal{C}_\bullet$ be an ∞ -category. An *object* of \mathcal{C} is an element of the set \mathcal{C}_0 of 0-simplices of \mathcal{C} . We will indicate that x is an object of \mathcal{C} by writing $x \in \mathcal{C}$.
- A *morphism* of \mathcal{C} is an element f of the set \mathcal{C}_1 of 1-simplices of \mathcal{C} . More precisely, we will say that f is a *morphism from x to y* if $\alpha_0^*(f) = x$ and $\alpha_1^*(f) = y$, where $\alpha_i : \{0\} \hookrightarrow \{0, 1\}$ denote the map given by $\alpha_i(0) = i$. We will often indicate that f is a morphism from x to y by writing $f : x \rightarrow y$.
- For any object $x \in \mathcal{C}$, there is an *identity morphism* id_x , given by $\beta^*(x)$ where $\beta : \{0, 1\} \rightarrow \{0\}$ is the unique map.
- Given a pair of morphisms $f, g : x \rightarrow y$ in \mathcal{C} , we say that f and g are *homotopic* if there exists a 2-simplex $\sigma \in \mathcal{C}_2$ whose faces are as indicated in the diagram



In this case, we will write $f \simeq g$, and we will say that σ is a *homotopy from f to g* . One can show that homotopy is an equivalence relation on the collection of morphisms from x to y .

- Given a pair of morphisms $f : x \rightarrow y$ and $g : y \rightarrow z$, it follows from Definition 13 that there exists a 2-simplex with boundary as indicated in the diagram



Definition 13 does *not* guarantee that the morphism h is unique. However, one can show that h is unique up to homotopy. We will generally abuse terminology and refer to h as the composition of f and g , and write $h = g \circ f$.

- Composition of morphisms in \mathcal{C} is associative up to homotopy. Consequently, we can define an ordinary category $\mathrm{h}\mathcal{C}$ as follows:
 - The objects of $\mathrm{h}\mathcal{C}$ are the objects of \mathcal{C} .
 - Given objects $x, y \in \mathcal{C}$, the set of morphisms from x to y in $\mathrm{h}\mathcal{C}$ is the set of equivalence classes (under the relation of homotopy) of morphisms from x to y in \mathcal{C} .
 - Given morphisms $[f] : x \rightarrow y$ and $[g] : y \rightarrow z$ in $\mathrm{h}\mathcal{C}$ represented by morphisms f and g in \mathcal{C} , we define $[g] \circ [f]$ to be the morphism from x to z in $\mathrm{h}\mathcal{C}$ given by the homotopy class of $g \circ f$.

We refer to $\mathrm{h}\mathcal{C}$ as the *homotopy category* of \mathcal{C} .

- We will say that a morphism f in \mathcal{C} is an *equivalence* if its image $[f]$ is an isomorphism in $\mathrm{h}\mathcal{C}$ (in other words, f is an equivalence if it admits an inverse up to homotopy). We say that two objects $x, y \in \mathcal{C}$ are *equivalent* if there exists an equivalence $f : x \rightarrow y$.

The theory of ∞ -categories allows us to treat topological spaces (via their singular simplicial sets) and ordinary categories (via the nerves) as examples of the same type of object. This is often very convenient.

Definition 17. Let \mathcal{C} and \mathcal{D} be ∞ -categories. A *functor* from \mathcal{C} to \mathcal{D} is a map of simplicial sets from \mathcal{C} to \mathcal{D} .

Remark 18. Let \mathcal{C} be an ∞ -category. The homotopy category of \mathcal{C} admits another characterization: it is universal among ordinary categories for which there exists a functor from \mathcal{C} to (the nerve of) $\mathrm{h}\mathcal{C}$.

Example 19. Let Z be a topological space and let \mathcal{C} be a category. Unwinding the definitions, we see that a functor from $\mathrm{Sing}(Z)_\bullet$ to $\mathrm{N}(\mathcal{C})_\bullet$ consists of the following data:

- (1) For each point $z \in Z$, an object $C_z \in \mathcal{C}$.
- (2) For every path $p : [0, 1] \rightarrow Z$, a morphism $\alpha_p : C_{p(0)} \rightarrow C_{p(1)}$, which is an identity morphism if the map p is constant.
- (3) For every continuous map $\Delta^2 \rightarrow Z$, which we write informally as

$$\begin{array}{ccc}
 & y & \\
 p \nearrow & & \searrow q \\
 x & \xrightarrow{r} & z,
 \end{array}$$

we have $\alpha_r = \alpha_q \circ \alpha_p$ (an equality of morphisms from C_x to C_z).

Here condition (3) encodes simultaneously the assumption that the map α_p depends only on the homotopy class of p , and that the construction $p \mapsto \alpha_p$ is compatible with concatenation of paths. Moreover, it follows from condition (3) that each of the maps α_p is an isomorphism (since every path is invertible up to homotopy). Consequently, we see that the data of a functor from $\mathrm{Sing}(Z)_\bullet$ into $\mathrm{N}(\mathcal{C})_\bullet$ recovers the classical notion of a *local system on Z with values in \mathcal{C}* .

One of the main advantages of working in the setting of ∞ -categories is that the collection of functors from one ∞ -category to another can easily be organized into a third ∞ -category.

Notation 20. For every integer $n \geq 0$, we let Δ^n denote the simplicial set given by the nerve of the linearly ordered set $\{0 < 1 < \dots < n\}$. We refer to Δ^n as the *standard n -simplex*. By definition, an m -simplex of Δ^n is given by a nondecreasing map $\{0, 1, \dots, m\} \rightarrow \{0, 1, \dots, n\}$.

Let X and Y be simplicial sets. We let $\text{Fun}(X, Y)$ denote the simplicial set of maps from X to Y . More precisely, $\text{Fun}(X, Y)$ is the simplicial set whose n -simplices are maps $\Delta^n \times X \rightarrow Y$ (more generally, giving a map of simplicial sets $Z \rightarrow \text{Fun}(X, Y)$ is equivalent to giving a map $Z \times X \rightarrow Y$).

One can show that if the simplicial set Y is an ∞ -category, then $\text{Fun}(X, Y)$ is also an ∞ -category (for any simplicial set X). Note that the objects of $\text{Fun}(X, Y)$ are functors from X to Y , in the sense of Definition 17. We will refer to $\text{Fun}(X, Y)$ as the *∞ -category of functors from X to Y* .

Example 21. Let \mathcal{C} and \mathcal{D} be ordinary categories. Then the simplicial set

$$\text{Fun}(N(\mathcal{C})_\bullet, N(\mathcal{D})_\bullet)$$

is isomorphic to the nerve of the category of functors from \mathcal{C} to \mathcal{D} . In particular, there is a bijection between the set of functors from \mathcal{C} to \mathcal{D} (in the sense of classical category theory) to the set of functors from $N(\mathcal{C})_\bullet$ to $N(\mathcal{D})_\bullet$ (in the sense of Definition 17).

Remark 22. It follows from Example 21 that no information is lost by passing from a category \mathcal{C} to the associated ∞ -category $N(\mathcal{C})$. For the remainder of this course, we will generally abuse notation by identifying each category \mathcal{C} with its nerve.

Example 23. Let Λ be a commutative ring and let $\text{Mod}_\Lambda = \{S_n\}_{n \geq 0}$ denote the simplicial set introduced in Construction 3. Then Mod_Λ is an ∞ -category, which we will refer to as the *derived ∞ -category of Λ -modules*. It can be regarded as an enhancement of the usual derived category $\mathcal{D}(\Lambda)$ of Λ -modules, in the sense that the homotopy category of Mod_Λ is equivalent to $\mathcal{D}(\Lambda)$ (in fact, the homotopy category of Mod_Λ is *isomorphic* to the category $\mathcal{D}'(\Lambda)$ defined above).

Notation 24. Let Λ be a commutative ring. For every integer n , the construction $M_* \mapsto H_n(M_*)$ determines a functor from the ∞ -category Mod_Λ to the ordinary abelian category of Λ -modules. We will say that an object $M_* \in \text{Mod}_\Lambda$ is *discrete* if $H_n(M_*) \simeq 0$ for $n \neq 0$. One can show that the construction $M_* \mapsto H_0(M_*)$ induces an equivalence from the ∞ -category of discrete objects of Mod_Λ to the ordinary category of Λ -modules. We will generally abuse notation by identifying the abelian category of Λ -modules with its inverse image under this equivalence. We will sometimes refer to Λ -modules as *discrete Λ -modules* or *ordinary Λ -modules*, to distinguish them from more general objects of Mod_Λ .

Remark 25. The ∞ -category Mod_Λ is, in many respects, easier to work with than the usual derived category $\mathcal{D}(\Lambda)$. For example, we have already mentioned that there is no functorial way to construct the cone of a morphism in $\mathcal{D}(\Lambda)$. However, Mod_Λ does not suffer from the same problem: the formation of cones is given by a functor $\text{Fun}(\Delta^1, \text{Mod}_\Lambda) \rightarrow \text{Mod}_\Lambda$.

The theory of ∞ -categories is a robust generalization of ordinary category theory. In particular, many important notions of ordinary category theory (adjoint functors, Kan extensions, Pro-objects and Ind-objects, ...) can be generalized to the setting of ∞ -categories in a natural way. We will make use of these notions throughout this course. For a detailed introduction (including complete definitions and proofs of the basic categorical facts we will need), we refer the reader to [2].

Notation 26. Let \mathcal{C} and \mathcal{D} be ∞ -categories. Throughout this course, we will often need to consider a *limit* or *colimit* of a functor $F : \mathcal{C} \rightarrow \mathcal{D}$. Roughly speaking, a *limit* of F is an object $D \in \mathcal{D}$ which is universal among those objects which are equipped with a family of morphisms $\{D \rightarrow F(C)\}_{C \in \mathcal{C}}$ (together with appropriate higher coherence data), and a *colimit* of F is an object $D' \in \mathcal{D}$ which is universal among those objects equipped with a compatible family of morphisms $\{F(C) \rightarrow D'\}_{C \in \mathcal{C}}$ (together with higher coherence data). We refer the reader to [2] for a more detailed discussion.

The limit and colimit of a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ are determined uniquely up to equivalence if they exist. We will generally abuse terminology by referring to *the* limit or colimit of a functor F , which we will denote by $\varprojlim_{C \in \mathcal{C}} F(C)$ and $\varinjlim_{C \in \mathcal{C}} F(C)$, respectively.

If F is given instead as a functor from the opposite ∞ -category \mathcal{C}^{op} to \mathcal{D} , we will generally denote a limit and colimit of F also by the notation

$$\varprojlim_{C \in \mathcal{C}} F(C) \quad \varinjlim_{C \in \mathcal{C}} F(C).$$

There is little danger of conflict between these notations, provided that it is clear from context whether the domain of the functor F is the ∞ -category \mathcal{C} or its opposite \mathcal{C}^{op} .

References

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