Nonabelian Poincare Duality in Algebraic Geometry (Lecture 9)

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In the previous lecture, we stated the following result:

Theorem 1 (Nonabelian Poincare Duality in Topology). *Let M be a manifold of dimension n and let (Y, y) be a pointed space which is $(n-1)$-connected. Then there exists a (homotopy) cosheaf $\mathcal{F}$ of spaces on the Ran space* $\mathrm{Ran}(M)$ *with the following property: for every collection of disjoint connected open sets $U_1, \ldots, U_k \subseteq M$, there is a canonical homotopy equivalence*

$$\mathcal{F}(\mathrm{Ran}(U_1, \ldots, U_k)) \simeq \prod_{1 \leq i \leq k} \mathrm{Map}_c(U_i, Y).$$

To get a feeling for what Theorem 1 is saying, it is useful to consider another example of space-valued cosheaves:

Example 2. Let $f : E \to B$ be a continuous function between topological spaces. Then the construction

$$U \mapsto f^{-1}(U)$$

determines a functor from the partially ordered set of open subsets of B to the category of topological spaces. This functor determines a cosheaf with values in the ∞-category of spaces: for example, for every pair of open sets $U, V \subseteq B$, the diagram of topological spaces

$$\begin{array}{ccc} f^{-1}(U \cap V) & \longrightarrow & f^{-1}(U) \\ \downarrow & & \downarrow \\ f^{-1}(V) & \longrightarrow & f^{-1}(U \cup V) \end{array}$$

is a homotopy pushout square (this is a mild strengthening of the existence of a Mayer-Vietoris sequence for computing the homology of $U \cup V$).

Let us imagine that every homotopy cosheaf on a topological space B arises via the construction of Example 2 (I do not know if this is actually true). Then Theorem 1 suggests that there should exist a continuous map of topological spaces

$$f : E \to \mathrm{Ran}(M)$$

with the property that for every collection of disjoint connected open sets $U_1, \ldots, U_k \subseteq M$, the inverse image $f^{-1} \mathrm{Ran}(U_1, \ldots, U_k)$ is homotopy equivalent to $\prod \mathrm{Map}_c(U_i, M)$. In particular, if M itself is connected, we deduce $E = f^{-1} \mathrm{Ran}(M)$ is homotopy equivalent to $\mathrm{Map}_c(M)$. We may therefore interpret Theorem 1 heuristically as suggesting that there should be a map $f : \mathrm{Map}_c(M) \to \mathrm{Ran}(M)$, whose inverse image over $\mathrm{Ran}(U_1, \ldots, U_k)$ consists of those functions from M into Y which are supported in a compact subset of $U_1 \cup \cdots \cup U_k$. In this lecture, we will describe how to realize this heuristic picture in the setting of algebraic geometry.

Fix an algebraically closed field k, an algebraic curve X over k, a smooth affine group scheme $\pi : G \to X$, and a prime number ℓ which is invertible in k. We have the following table of analogies with the topological setting discussed in the previous lecture:

Abelian Cohomology	Nonabelian Cohomology	Algebraic Geometry
Manifold M	Manifold M	Algebraic Curve X
Abelian group A	Pointed topological space (Y, y)	G (or BG)
$C^*_c(M; A)$	$\mathrm{Map}_c(M, Y)$	$\mathrm{Bun}_G(X)$
Open disk $U \subseteq M$	Open disk $U \subseteq M$	Formal completion of X at a point $x \in X$
$C^*_c(U; A)$	$\mathrm{Map}_c(U, Y)$	G-bundles on X trivialized on $X - \{x\}$
	$\mathrm{Ran}(M)$	$\mathrm{Ran}(X)$

Here $\mathrm{Ran}(X)$ denotes the prestack classifying nonempty finite sets with a map to X, which we introduced in Lecture 7. We now consider a slightly more elaborate version of this definition:

Definition 3. For every k-algebra R, we let X_R denote the relative curve $\mathrm{Spec}\, R \times_{\mathrm{Spec}\, k} X$. If S is a subset of $X(R)$, then each element $s \in S$ can be regarded as a section of the projection map $X_R \to \mathrm{Spec}\, R$. We let $|S|$ denote the union of the images of these sections (which we regard as a closed subset of X_R).

We define a category $\mathrm{Ran}_G(X)$ as follows:

- An object of $\mathrm{Ran}_G(X)$ consists of a quintuple $(R, S, \mu, \mathcal{P}, \gamma)$ where R is a finitely generated k-algebra, S is a nonempty finite set, $\mu : S \to X(R)$ is a map of sets, and $\mathcal{P}$ is a G-bundle on the relative curve X_R, S is a nonempty finite subset of $X(R)$, and γ is a trivalization of $\mathcal{P}$ over the open set $X_R - |\mu(S)|$.
- A morphism from $(R, S, \mu, \mathcal{P}, \gamma)$ to $(R', S', \mu', \mathcal{P}', \gamma')$ is a morphism $(\phi, \alpha) : (R, \mathcal{P}) \to (R', \mathcal{P}')$ in the category $\mathrm{Bun}_G(X)$ together with a surjection of finite sets $S \to S'$ such that the diagram

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ X(R) & \longrightarrow & X(R') \end{array}$$

  commutes and the isomorphism α carries γ (which is a trivialization of $\mathcal{P}\,|_{X_R - |S|}$) to γ' (which is a trivialization of $\mathcal{P}'\,|_{X_{R'} - |S'|} = \mathrm{Spec}\, R' \times_{\mathrm{Spec}\, R} \mathcal{P}\,|_{X_R - |S|}$).

The construction $(R, S, \mu, \mathcal{P}, \gamma) \mapsto R$ determines a forgetful functor $\mathrm{Ran}_G(X) \to \mathrm{Ring}_k$, which exhibits $\mathrm{Ran}_G(X)$ as a prestack. We will sometimes refer to $\mathrm{Ran}_G(X)$ as the *Beilinson-Drinfeld Grassmannian of* G.

We can think of $\mathrm{Ran}_G(X)$ as parametrizing moduli of G-bundles on X which are equipped with a trivialization outside of some (specified) finite subset of X. Roughly speaking, it can be viewed as an amalgamation over all finite subsets $S \subseteq X$ of the moduli spaces of G-bundles which are supported in a small neighborhood of S. We may therefore regard $\mathrm{Ran}_G(X)$ as an algebro-geometric incarnation of the homotopy colimit

$$\varinjlim_{U_1, \ldots U_k} \mathrm{Map}_c(U_1, BG) \times \cdots \times \mathrm{Map}_c(U_k, BG)$$

appearing in the previous lecture (at least when G is a constant group). Note that there is an evident projection map $\rho : \mathrm{Ran}_G(X) \to \mathrm{Bun}_G(X)$, given on objects by $(R, S, \mu, \mathcal{P}, \gamma) \mapsto (R, \mathcal{P})$. The first main result of this course is the following:

Theorem 4 (Nonabelian Poincare Duality)**.** *Suppose that the generic fiber of G is semisimple and simply connected. Then the map ρ induces an isomorphism on ℓ-adic homology*

$$\mathrm{H}_*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell) \to \mathrm{H}_*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell).$$

Remark 5. The hypothesis that the generic fiber of G be semisimple and simply connected can be relaxed somewhat. In these lectures, we will often make the simplifying assumption that the group scheme G is constant: in this case, our proof will work for an arbitrary connected affine group (note that the topological analogue of Theorem 4, which we discussed in the previous lecture, requires only that the classifying space BG is 1-connected, or that G is connected).

Remark 6. It follows formally from Theorem 4 that the map $\rho : \mathrm{Ran}_G(X) \to \mathrm{Bun}_G(X)$ induces isomorphisms on homology and cohomology with coefficients in $\mathbf{Z}_\ell$, $\mathbf{Q}_\ell$, and $\mathbf{Z}/\ell^d\mathbf{Z}$.

Theorem 4 allows us to reduce the problem of understanding the cohomology of $\mathrm{Bun}_G(X)$ to the problem of understanding the cohomology of $\mathrm{Ran}_G(X)$. Note that there is an evident forgetful functor $\theta : \mathrm{Ran}_G(X) \to \mathrm{Ran}(X)$, given on objects by $(R, S, \mu, \mathcal{P}, \gamma) \mapsto (R, S, \mu)$. We can therefore identify the cohomology of $\mathrm{Ran}_G(X)$ (with coefficients in the constant sheaf $\underline{\mathbf{Z}}_\ell$) with the cohomology of $\mathrm{Ran}(X)$ with coefficients in the (derived) direct image sheaf $\theta_* \underline{\mathbf{Z}}_\ell$. This is useful because it breaks the problem up into two parts: understanding the topology of $\mathrm{Ran}(X)$ (which is independent of the group scheme G) and understanding the structure of the sheaf $\theta_* \underline{\mathbf{Z}}_\ell$ (which depends only on the *local* behavior of the group scheme G).

To understand the structure of the Beilinson-Drinfeld Grassmannian a bit better, it is useful to first think about the fibers of the map θ. Fix a closed point $x \in X$, which we can identify with a k-valued point of $\mathrm{Ran}(X)$ (via the inclusion $\{x\} \hookrightarrow X(k)$). Let $\mathrm{Gr}_G^{\{x\}}$ denote the fiber product $\mathrm{Spec}\, k \times_{\mathrm{Ran}(X)} \mathrm{Ran}_G(X)$. Unwinding the definitions, we see that $\mathrm{Gr}_G^{\{x\}}$ is a prestack whose objects are given by triples $(R, \mathcal{P}, \gamma)$, where R is a finitely generated k-algebra, $\mathcal{P}$ is a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}$ over the open set $(X - \{x\})_R$. Note that this is actually a prestack in sets: any automorphism of a G-bundle $\mathcal{P}$ on X_R which is the identity on the dense open set $(X - \{x\})_R$ must be the identity over all of X_R.

Let $\mathcal{O}_x \simeq k[[t]]$ denote the completed local ring of X at the point x and let $K_x \simeq k((x))$ denote its function field. To give a G-bundle $\mathcal{P}$ on the curve X, one must supply a G-bundle $\mathcal{P}_0$ over the open set $X - \{x\}$, a G-bundle $\mathcal{P}_1$ over the formal disk $\mathrm{Spec}\, \mathcal{O}_x$, and an isomorphism between the restrictions of $\mathcal{P}_0$ and $\mathcal{P}_1$ to $\mathrm{Spec}\, K_x$. In particular, we can identify k-valued points of $\mathrm{Gr}_G^{\{x\}}$ with pairs $(\mathcal{P}_1, \gamma)$, where $\mathcal{P}_1$ is a G-bundle on $\mathrm{Spec}\, \mathcal{O}_x$ and γ is a trivialization of $\mathcal{P}_1$ over $\mathrm{Spec}\, K_x$. The G-bundle $\mathcal{P}_1$ is automatically trivial (since k is algebraically closed and G is smooth over X). If we fix a trivialization of $\mathcal{P}_1$, then we can identify γ with an element of the group $G(K_x)$. This element depends on our choice of trivialization of $\mathcal{P}_1$, and is therefore ambiguous up to multiplication by elements of $G(\mathcal{O}_x)$. This construction yields a bijection

$$\mathrm{Gr}_G^{\{x\}}(k) \simeq G(K_x)/G(\mathcal{O}_x) = G(k((t)))/G(k[[t]]).$$

We may describe the situation roughly by saying that the $G(k((t)))/G(k[[t]])$ is equipped with an algebro-geometric structure: that is, there is a notion of algebraic map from $\mathrm{Spec}\, R$ to $G(k((t)))/G(k[[t]])$ (such maps are parametrized by the set $\mathrm{Gr}_G^{\{x\}}(R)$).

The prestack $\mathrm{Gr}_G^{\{x\}}$ is not representable by a scheme. However, it is not far off:

Fact 7. *The prestack $\mathrm{Gr}_G^{\{x\}}$ is equivalent to the direct limit of a sequence*

$$Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{f_2} \cdots$$

where each Y_i is a quasi-projective k-scheme and each f_i is a closed immersion. If G is reductive at the point x, then each Y_i can be chosen to be a projective k-scheme.

Example 8. Let us say that a subset $L \subseteq K_x^n$ is a *lattice* if it is a free $\mathcal{O}_x$-module of rank n. The group $\mathrm{GL}_n(K_x)$ acts on the set of lattices, and the stabilizer group of the standard lattice $\mathcal{O}_x^n$ is the general linear group $\mathrm{GL}_n(\mathcal{O}_x)$. We may therefore identify

$$\mathrm{Gr}_{\mathrm{GL}_n}^{\{x\}}(k) \simeq \mathrm{GL}_n(K_x)/\,\mathrm{GL}_n(\mathcal{O}_x)$$

with the set of all lattices in K_x^n.

For each $d \geq 0$, let Z_d denote the set of all lattices $L \subseteq K_x^n$ such that

$$t^d\,\mathcal{O}_x^n \subseteq L \subseteq t^{-d}\,\mathcal{O}_x^n\,.$$

Such lattices are uniquely determined by the image of the quotient map

$$L/t^d\,\mathcal{O}_x^n \hookrightarrow V = t^{-d}\,\mathcal{O}_x^n\,/t^d\,\mathcal{O}_x^n\,.$$

We may therefore identify Z_d with a subset of the disjoint union

$$\amalg_{0 \leq q \leq dn}\,\mathrm{Gr}_q(V)(k),$$

where $\mathrm{Gr}_q(V)$ denotes the Grassmannian parametrizing q-dimensional subspaces of V (and $\mathrm{Gr}_q(V)(k)$ denotes its set of k-valued points). One can show that this subset is Zariski-closed (it is the set of all subspaces which are invariant under the action of $\mathcal{O}_x$), so that Z_d can be identified with the set of k-valued points of a closed subscheme of $\amalg_{0 \leq q \leq dn}\,\mathrm{Gr}_q(V)$ (which is a projective k-scheme). We may therefore write

$$\mathrm{Gr}_{\mathrm{GL}_n}^{\{x\}}(k) = \bigcup_{d \geq 0} Z_d$$

as the k-valued points of the union of an increasing family of projective k-schemes. In other words, $\mathrm{Gr}_{\mathrm{GL}_n}^{\{x\}}$ behaves in some respect like an infinite-dimensional projective variety.