Three Descriptions of the Cohomology of $\mathrm{Bun}_G(X)$ (Lecture 4)

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Let k be an algebraically closed field, let X be a algebraic curve over k (always assumed to be smooth and complete), and let G be a smooth affine group scheme over X. We let $\mathrm{Bun}_G(X)$ denote the moduli stack of G-bundles on X (denoted by Bun_G in the previous lecture), and let ℓ denote a prime number which is invertible in k. Our main goal in this course is to give a convenient description of the ℓ-adic cohomology ring $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$. In the special case where G and X are defined over some finite field $\mathbf{F}_q$, this will allow us to compute the trace of φ^{-1} on $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$ (where φ denotes the geometric Frobenius morphism from $\mathrm{Bun}_G(X)$ to itself), which we will use to prove Weil's conjecture following the outline provided in the last lecture. However, the problem of describing $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$ makes sense over an *arbitrary* algebraically closed field k. In this lecture, we will specialize to the case $k = \mathbf{C}$ and describe several topological approaches to the problem (which we will later adapt to the setting of algebraic geometry). In this case, we do not need to work ℓ-adically: the algebraic stack $\mathrm{Bun}_G(X)$ has a well-defined homotopy type (namely, the homotopy type of the associated analytic stack). We will abuse notation by identifying X with the compact Riemann surface $X(\mathbf{C})$.

To simplify the discussion, let us assume that the group scheme G is semisimple and simply connected at each point. Fix a G-bundle $\mathcal{P}_{\mathrm{sm}}$ in the category of smooth manifolds. The tangent bundle of $\mathcal{P}_{\mathrm{sm}}$ is a G-equivariant vector bundle on $\mathcal{P}_{\mathrm{sm}}$, and can therefore be written the pullback of a smooth vector bundle $\mathcal{E}$ on X. This vector bundle fits into an exact sequence

$$0 \to \mathcal{E}_0 \to \mathcal{E} \to T_X \to 0, \tag{1}$$

where $\mathcal{E}_0$ denotes the vector bundle associated by $\mathcal{P}_{\mathrm{sm}}$ to the adjoint representation of G. In particular, we can regard $\mathcal{E}_0$ as a complex vector bundle on X. A $\overline{\partial}$*-connection* on $\mathcal{P}_{\mathrm{sm}}$ is a choice of complex structure on the vector bundle $\mathcal{E}$ for which (1) is an exact sequence of *complex* vector bundles on X. Let Ω denote the collection of all $\overline{\partial}$-connections on X. Then Ω can be regarded as a torsor for the infinite-dimensional vector space of $\mathbf{C}$-antilinear bundle maps from T_X into $\mathcal{E}_0$: in particular, it is an infinite-dimensional affine space, and therefore contractible.

Proposition 1. *Let $\mathcal{G} = \mathrm{Aut}(\mathcal{P}_{\mathrm{sm}})$ denote the group of all automorphisms of the smooth G-bundle $\mathcal{P}_{\mathrm{sm}}$. Then the moduli stack $\mathrm{Bun}_G(X)$ can be identified, as a differentiable stack, with the quotient of the contractible space Ω by the action of the $\mathcal{G}$. In particular, $\mathrm{Bun}_G(X)$ has the homotopy type of the classifying space $\mathrm{B}\mathcal{G}$.*

Sketch. Every G-bundle $\rho : \mathcal{P} \to X$ is a fiber bundle with simply connected fibers, and is therefore trivial in the category of smooth G-bundles (since X is real manifold of dimension 2). In particular, for every complex-analytic G-bundle $\mathcal{P}$ on X, we can choose an isomorphism of smooth G-bundles $\alpha : \mathcal{P}_{\mathrm{sm}} \to \mathcal{P}$. We can identify isomorphism classes of pairs $(\mathcal{P}, \alpha)$ with complex-analytic structures on the bundle $\mathcal{P}_{\mathrm{sm}}$; since X has dimension ≤ 1, these are in bijection with points of Ω. Then $\mathcal{G}$ acts on the space Ω, and the homotopy quotient of Ω by $\mathcal{G}$ classifies complex-analytic G-bundles on X. Since X is a projective algebraic variety, the category of complex-analytic G-bundles on X is equivalent to the category of algebraic vector bundles on X. (For a related discussion, see [1].) □

Remark 2. The argument we sketched above really proves that the groupoid $\mathrm{Bun}_G(X)(\mathbf{C})$ of $\mathbf{C}$-valued points of $\mathrm{Bun}_G(X)$ can be identified with the groupoid quotient of Ω (regarded as a set) by $\mathcal{G}$ (regarded as a discrete group). To formulate a stronger claim, we would need to be more precise about the procedure which associates a homotopy type to an algebraic stack over $\mathbf{C}$. A reader who is concerned with this technical point should feel free to take Principle 1 as a *definition* of the homotopy type of $\mathrm{Bun}_G(X)$.

Warning 3. The validity of Principle 1 relies crucially on the fact that X is an algebraic curve. If X is a smooth projective variety of higher dimension, then smooth G-bundles on X need not be trivial, and $\overline{\partial}$-connections on a smooth G-bundle $\mathcal{P}_{\mathrm{sm}}$ need not be integrable. Consequently, the homotopy type of $\mathrm{Bun}_G(X)$ is not so easy to describe.

Note that since the G-bundle $\mathcal{P}_{\mathrm{sm}}$ is trivial, we can identify the gauge group $\mathcal{G}$ with the space of all smooth sections of the projection map $G \to X$. We would like to use this information to describe the homotopy type of the classifying space of $\mathrm{B}\mathcal{G}$ in terms of the individual classifying spaces $\{\mathrm{BG}_x\}_{x \in X}$. We next outline three approaches to this problem: the first allows us to express $\mathrm{H}^*(\mathrm{B}\mathcal{G}; \mathbf{Q})$ as the cohomology of a certain differential graded Lie algebra (Theorem 4), while the remaining two express $\mathrm{H}^*(\mathrm{B}\mathcal{G}; \mathbf{Q})$ and $\mathrm{H}_*(\mathrm{B}\mathcal{G}; \mathbf{Q})$ as the homology of certain factorization algebras on X (Theorems 9 and 13).

1 First Approach: Rational Homotopy Theory

Let H be a path-connected topological group. Then the homology $\mathrm{H}_*(H; \mathbf{Q})$ has the structure of a co-commutative Hopf algebra: the multiplication on $\mathrm{H}_*(H; \mathbf{Q})$ is given by pushforward along the product map $H \times H \to H$, and the comultiplication on $\mathrm{H}_*(H; \mathbf{Q})$ is given by pushforward along the diagonal map $\delta : H \to H \times H$. With more effort, one can construct an analogue of this Hopf algebra structure at the level of chains, rather than homology. More precisely, Quillen's work on rational homotopy theory gives a functorial procedure for associating to each topological group H a differential graded Lie algebra $\mathfrak{g}(H)$ (defined over the field of rational numbers) with the following properties:

(*a*) Let $\mathrm{H}_*(\mathfrak{g}(H))$ denote the homology groups of the underlying chain complex of $\mathfrak{g}(H)$. Then we have a canonical isomorphism

$$\mathbf{Q} \otimes \pi_* H \simeq \mathrm{H}_*(\mathfrak{g}(H)).$$

Under this isomorphism, the Whitehead product on $\pi_{*+1}\, \mathrm{BH} \simeq \pi_* H$ corresponds to the Lie bracket on $\mathrm{H}_*(\mathfrak{g}(H))$.

(*b*) The singular chain complex $C_*(H; \mathbf{Q})$ is canonically quasi-isomorphic to the universal enveloping algebra $U(\mathfrak{g}(H))$. This quasi-isomorphism induces a Hopf algebra isomorphism

$$\mathrm{H}_*(U(\mathfrak{g}(H))) \simeq \mathrm{H}_*(H; \mathbf{Q}).$$

(*c*) The differential graded Lie algebra $\mathfrak{g}(H)$ is a complete invariant of the rational homotopy type of the classifying space BH. More precisely, from $\mathfrak{g}(H)$ one can functorially construct a pointed topological space Z for which there exists a pointed map $\mathrm{BH} \to Z$ which induces an isomorphism on rational cohomology. In particular, the cohomology ring $\mathrm{H}^*(\mathrm{BH}; \mathbf{Q})$ can be functorially recovered as the *Lie algebra cohomology* of $\mathfrak{g}(H)$.

Let us now apply the above reasoning to our situation. For every open subset $U \subseteq X$, let $\mathcal{G}_U$ denote the (topological) group of all smooth sections of the projection map $G \times_X U \to U$, and let $\mathfrak{g}(\mathcal{G}_U)$ be the associated differential graded Lie algebras. The construction $U \mapsto \mathfrak{g}(\mathcal{G}_U)$ is contravariantly functorial in U. For each integer n, let $\mathcal{F}_n$ denote the presheaf of rational vector spaces on X given by $\mathcal{F}_n(U) = \mathfrak{g}(\mathcal{G}_U)_n$, and let $\overline{\mathcal{F}}_n$ be the associated sheaf. Ignoring the Lie algebra structures on the differential graded Lie algebras $\mathfrak{g}(\mathcal{G}_U)$ and remember only the underlying chain complexes, we obtain a chain complex of presheaves

$$\cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{F}_{-1} \to \mathcal{F}_{-2} \to \cdots,$$

hence a chain complex of sheaves

$$\cdots \to \overline{\mathcal{F}}_2 \to \overline{\mathcal{F}}_1 \to \overline{\mathcal{F}}_0 \to \overline{\mathcal{F}}_{-1} \to \overline{\mathcal{F}}_{-2} \to \cdots.$$

In this language, we can formulate a local-to-global principle as follows:

Theorem 4. *The canonical map*

$$\mathfrak{g}(\mathcal{G}) = \Gamma(X; \mathcal{F}_*) \to \Gamma(X; \overline{\mathcal{F}}_*) \to R\Gamma(X; \overline{\mathcal{F}}_*)$$

is a quasi-isomorphism of differential graded Lie algebras. In other words, the cohomology groups of the differential graded Lie algebra $\mathfrak{g}(\mathcal{G})$ can be identified with the hypercohomology groups of the chain complex $\overline{\mathcal{F}}_*$ of sheaves on X.

Proof. This follows from the compatibility of the construction $H \mapsto \mathfrak{g}(H)$ with (suitable) homotopy inverse limits. □

Remark 5. Fix a point $x \in X$. If $U \subseteq X$ is an open disk containing x, then evaluation at x induces a homotopy equivalence of topological groups $\mathcal{G}_U \to G_x$. Passing to the direct limit, we obtain a quasi-isomorphism of chain complexes $\mathcal{F}_{*,x} \simeq \overline{\mathcal{F}}_{*,x} \to \mathfrak{g}(G_x)$. In particular, the nth homology of the complex $\overline{\mathcal{F}}_*$ is a locally constant sheaf on X. Theorem 4 then supplies a convergent spectral sequence

$$\mathrm{H}^s(X; \mathbf{Q} \otimes \pi_t(G_\bullet)) \Rightarrow \mathbf{Q} \otimes \pi_{t-s}\, \mathcal{G},$$

where $\mathbf{Q} \otimes \pi_t(G_\bullet)$ denotes the local system of rational vector spaces on X given by $x \mapsto \mathbf{Q} \otimes \pi_t(G_x)$.

Example 6 (Atiyah-Bott)**.** Suppose that G is constant: that is, it is the product of X with a simply connected semisimple algebraic group G_0 over $\mathbf{C}$. In this case, the chain complex $\overline{\mathcal{F}}_*$ is quasi-isomorphic to the chain complex of constant sheaves with value $\mathfrak{g}(G_0)$. In this case, Theorem 4 supplies a quasi-isomorphism

$$\mathfrak{g}(\mathcal{G}) \simeq C^*(X; \mathbf{Q}) \otimes_{\mathbf{Q}} \mathfrak{g}(G_0)$$

.

The rational cohomology of the classifying space BG_0 is isomorphic to a polynomial ring $\mathbf{Q}[t_1, \ldots, t_r]$, where r is the rank of the semisimple algebraic group G_0 and each t_i is a homogeneous element of $\mathrm{H}^*(\mathrm{BG}_0; \mathbf{Q})$ of some even degree e_i. From this, one can deduce that the differential graded Lie algebra $\mathfrak{g}(G_0)$ is *formal*: that is, it is quasi-isomorphic to a graded vector space V on generators $t_i^\vee$ of (homological) degree $e_i - 1$, where the differential and the Lie bracket vanish. It follows that $\mathfrak{g}(\mathcal{G})$ is quasi-isomorphic to the tensor product $\mathrm{H}^*(X; \mathbf{Q}) \otimes_{\mathbf{Q}} V$, where the differential and Lie bracket vanish. From this, one can deduce that $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q})$ is isomorphic to a (graded) symmetric algebra on the graded vector space $\mathrm{H}_*(X; \mathbf{Q}) \otimes_{\mathbf{Q}} V^\vee[-1]$. In other words, $\mathrm{H}^*(\mathcal{M}; \mathbf{Q})$ is a tensor product of a polynomial ring on $2r$ generators in even degrees with an exterior algebra on $2gr$ generators in odd degrees. We refer the reader to [1] for more details.

2 Second Approach: Factorization Homology

Theorem 4 asserts that that the differential graded Lie algebra $\mathfrak{g}(\mathcal{G})$ can be recovered as the hypercohomology of a "local system" of differential graded Lie algebras given by $x \mapsto \mathfrak{g}(G_x)$. Roughly speaking, this reflects the idea that the gauge group $\mathcal{G}$ can be identified with a "continuous product" of the groups G_x, and that the construction $H \mapsto \mathfrak{g}(H)$ is compatible with "continuous products" (at least in good cases).

Our ultimate goal is to formulate a local-to-global principle which will allow us to compute the rational cohomology ring $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}) \simeq \mathrm{H}^*(\mathrm{B}\mathcal{G}; \mathbf{Q})$. It is possible to formulate such a principle directly, without making a detour through the theory of differential graded Lie algebras. However, the basic mechanism of the local-to-global principle takes a more complicated form.

Definition 7. For each open set $U \subseteq X$, let $\mathcal{B}(U)$ denote the rational cochain complex $C^*(B\,\mathcal{G}_U; \mathbf{Q})$. Then the construction $U \mapsto \mathcal{B}(U)$ determines a covariant functor from the partially ordered set of open subsets of X to the category of chain complexes of rational vector spaces.

Let $\mathcal{U}$ denote the collection of all open subsets of X which can be written as a disjoint union of disks. We let $\int \mathcal{B}$ denote a homotopy colimit of the diagram $\{\mathcal{B}(U)\}_{U \in \mathcal{U}}$ (in the category of chain complexes of rational vector spaces). We refer to the homology of the chain complex $\int \mathcal{B}$ as the *factorization homology* of $\mathcal{B}$.

Example 8. Suppose that $U \subseteq X$ is an open set which can be written as a disjoint union $U_1 \cup \cdots \cup U_n$, where each U_i is an open disk. Choose a point $x_i \in U_i$ for $1 \leq i \leq n$. Then $\mathcal{G}_U$ is homeomorphic to a product $\prod_{1 \leq i \leq n} \mathcal{G}_{U_i}$, and evaluation at the points x_i determine homotopy equivalences $\mathcal{G}_{U_i} \to G_{x_i}$. Consequently, there is a canonical quasi-isomorphism of chain complexes

$$\begin{aligned} \bigotimes_{1 \leq i \leq n} C^*(\mathrm{BG}_{x_i}; \mathbf{Q}) &\stackrel{\sim}{\to} \bigotimes_{1 \leq i \leq n} C^*(B\,\mathcal{G}_{U_i}; \mathbf{Q}) \\ &\stackrel{\sim}{\to} C^*(\mathrm{B}\mathcal{G}_U; \mathbf{Q}) \\ &= \mathcal{B}(U). \end{aligned}$$

In other words, each term in the diagram $\{\mathcal{B}(U)\}_{U \in \mathcal{U}}$ can be identified with a tensor product

$$\bigotimes_{x \in S} C^*(\mathrm{BG}_x; \mathbf{Q}),$$

where S is some finite subset of X. We can therefore think of the factorization homology $\int \mathcal{B}$ as a kind of continuous tensor product $\bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Q})$. We refer the reader to [3] for more details.

We can now formulate a second local-to-global principle for describing the cohomology of $\mathrm{Bun}_G(X)$:

Theorem 9. *If the fibers of G are semisimple and simply connected, then the canonical map*

$$\int \mathcal{B} = \mathrm{hocolim}_{U \in \mathcal{U}}\, \mathcal{B}(U) \to \mathcal{B}(X) = C^*(\mathrm{B}\mathcal{G}; \mathbf{Q}) = C^*(\mathrm{Bun}_G(X); \mathbf{Q})$$

is a quasi-isomorphism. In other words, we can identify the cohomology of the moduli stack $\mathrm{Bun}_G(X)$ *with the factorization homology of* $\mathcal{B}$.

3 Third Approach: Nonabelian Poincare Duality

The local-to-global principle expressed by Theorem 9 is based on the idea of approximating the moduli stack $\mathrm{Bun}_G(X) \simeq \mathrm{B}\mathcal{G}$ "from the right". For any finite set $S \subseteq X$, evaluation at the points of S defines a map of classifying spaces

$$\mathrm{B}\mathcal{G} \to \prod_{x \in S} \mathrm{BG}_x,$$

hence a map of cochain complexes

$$\mu_S : \bigotimes_{x \in S} C^*(\mathrm{BG}_x; \mathbf{Q}) \to C^*(\mathrm{B}\mathcal{G}; \mathbf{Q}).$$

Roughly speaking, Theorem 9 asserts that if we allow S to vary continuously over all finite subsets of X, then we can use these maps to recover the chain complex $C^*(\mathrm{Bun}_G(X); \mathbf{Q})$ up to quasi-isomorphism. We now explore an parallel approach, which is based on the idea of realizing $\mathrm{B}\mathcal{G}$ as direct limit, rather than an inverse limit.

Notation 10. For each open set $U \subseteq X$, let $\mathcal{G}_U^c$ denote the subgroup of $\mathcal{G}$ consisting of those automorphisms of $\mathcal{P}_{\mathrm{sm}}$ which are the identity outside of a compact subset of U, and let $\mathcal{A}(U)$ denote the chain complex $C_*(\mathrm{B}\mathcal{G}_U^c; \mathbf{Q})$. Note that $\mathcal{G}_U^c \subseteq \mathcal{G}_V^c$ whenever $U \subseteq V$, so that we can regard the construction $U \mapsto \mathcal{A}(U)$ is a covariant functor from the partially ordered set of open subsets of X to the category of chain complexes.

Let $\mathcal{U}$ denote the collection of all open subsets of X which can be written as a disjoint union of disks. We let $\int \mathcal{A}$ denote a homotopy colimit of the diagram $\{\mathcal{A}(U)\}_{U \in \mathcal{U}}$ (in the category of chain complexes of rational vector spaces). We refer to the homology of the chain complex $\int \mathcal{A}$ as the *factorization homology* of $\mathcal{A}$.

Example 11. Let $U \subseteq X$ be an open disk containing a point $x \in X$. The $U \times_X G$ is diffeomorphic to a product $U \times G_x$, so that that $\mathcal{G}_U^c$ can be identified with the space of compactly supported maps from U into G_x. A choice of homeomorphism $U \simeq \mathbb{R}^2$ then determines a homotopy equivalence of $\mathcal{G}_U^c$ with the two-fold loop space $\Omega^2(G_x)$, so that $\mathrm{B}\mathcal{G}_U^c$ can be identified with $\Omega^2(\mathrm{BG}_x) \simeq \Omega(G_x)$.

More generally, if U can be written as a disjoint union of disks $U_1 \cup \cdots \cup U_n$ containing points $x_i \in U_i$, then $\mathcal{G}_U^c$ is homeomorphic to a product $\prod_{1 \leq i \leq n} \mathcal{G}_{U_i}^c$, so we obtain a quasi-isomorphism of chain complexes

$$\begin{aligned} \bigotimes_{1 \leq i \leq n} C_*(\Omega^2 \, \mathrm{BG}_{x_i}; \mathbf{Q}) &\simeq \bigotimes_{1 \leq i \leq n} C_*(\mathrm{B}\mathcal{G}_{U_i}^c; \mathbf{Q}) \\ &\simeq C_*(\mathrm{B}\mathcal{G}_U^c; \mathbf{Q}) \\ &= \mathcal{A}(U). \end{aligned}$$

In other words, each term in the diagram $\{\mathcal{A}(U)\}_{U \in \mathcal{U}}$ can be identified with a tensor product

$$\bigotimes_{x \in S} C_*(\Omega^2(\mathrm{BG}_x); \mathbf{Q}),$$

where S is some finite subset of X. We can therefore think of the factorization homology $\int \mathcal{A}$ as a kind of continuous tensor product $\bigotimes_{x \in X} C_*(\Omega^2(\mathrm{BG}_x); \mathbf{Q})$.

Remark 12. The double loop space $\Omega^2(\mathrm{BG}_x)$ is homotopy equivalent to quotient $G(K_x)/G(\mathcal{O}_x)$, where $\mathcal{O}_x$ denotes the completed local ring of X at x, and K_x denotes its fraction field. We will denote this quotient by Gr_G^x and refer to it as the *affine Grassmannian* of the group G at the point x. This paper depends crucially on the fact that Gr_G^x admits an algebro-geometric incarnation (as the direct limit of a sequence of algebraic varieties) and can be defined over ground fields different from $\mathbf{C}$.

We have the following analogue of Theorem 9:

Theorem 13 (Nonabelian Poincare Duality)**.** *The canonical map*

$$\int \mathcal{A} = \mathrm{hocolim}_{U \in \mathcal{U}} \, \mathcal{A}(U) \to \mathcal{A}(X) = C_*(\mathrm{B}\mathcal{G}; \mathbf{Q}) = C_*(\mathrm{Bun}_G(X); \mathbf{Q})$$

is a quasi-isomorphism. In other words, we can identify the homology of the moduli stack $\mathrm{Bun}_G(X)$ *with the factorization homology of* $\mathcal{A}$*.*

Remark 14. Theorem 13 can be regarded as version of Poincare duality for the manifold X with coefficients in the nonabelian group G. We will explain this idea in more detail in §**??**.

Let us now outline the relationship between Theorems 4, 9, and 13.

- Theorem 4 is the weakest of the three results. It only gives information about the rational homotopy type of the moduli stack $\mathrm{Bun}_G(X)$, while Theorems 9 and 13 remain valid with integral coefficients. In fact, Theorem 13 is even true at the “unstable” level: that is, it gives a procedure for reconstructing the space $\mathrm{B}\mathcal{G}$ itself, rather than just the singular chain complex of $\mathrm{B}\mathcal{G}$). However, Theorem 4 gives information in a form which is most amenable to further calculation, since it articulates a local-to-global principle using the familiar language of sheaf cohomology, rather than the comparatively exotic language of factorization homology.

- Theorem 13 can also be regarded as the strongest of the three results because it requires the weakest hypotheses: if it is formulated correctly, we only need to assume that the fibers of the map $G \to X$ are connected, rather than simply connected.

- Theorems 13 and 9 can be regarded as duals of one another. More precisely, the construction $x \mapsto C^*(\mathrm{BG}_x; \mathbf{Q})$ determines a factorization algebra on X which is *Koszul dual* to the factorization algebra $x \mapsto C_*(\Omega^2 \, \mathrm{BG}_x; \mathbf{Q})$. Using this duality, one can construct a duality pairing between the chain complexes $\int \mathcal{A}$ and $\int \mathcal{B}$, which identifies each with the $\mathbf{Q}$-linear dual of the other (under the assumption that the fibers of G are simply connected).

- Theorems 4 and 9 can also be regarded as duals of one another, but in a different sense. Namely, each of the differential graded Lie algebras $\mathfrak{g}(G_x)$ can be regarded as the Koszul dual of $C^*(\mathrm{BG}_x; \mathbf{Q})$, which we regard as an E_∞-algebra over $\mathbf{Q}$. One can exploit this to deduce Theorem 4 from Theorem 9 and vice versa.

We can now outline the contents of this course. Our first step will be to formulate and prove an ℓ-adic version of Proposition 13, which is valid over an arbitrary algebraically closed ground field k. The proof is based on a calculation of the homology of the "space" of generic trivializations of an arbitrary G-bundle $\mathcal{P}$. Our next step will be to give a precise articulation of duality between Propositions 9 and 13. The basic mechanism is a local duality calculation which allows us to recover the ℓ-adic cohomology of a classifying space BG_x from the ℓ-adic cohomology of $\Omega^2 BG_x$ (which arises in its algebro-geometric incarnation as the *affine Grassmannian* of the group G). We will then use this result to deduce an ℓ-adic version of Proposition 4. Combining this with the usual Grothendieck-Lefschetz trace formula, this supplies information about the action of Frobenius on the differential graded Lie algebra $\mathfrak{g}_*(\mathcal{M})$, which will in turn yield the desired information about the action of Frobenius on $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$.

Many steps in the preceding outline will require us to carry out sophisticated chain-level constructions which are difficult to describe using the language of classical homological algebra. In the next lecture, we give a brief introduction to the theory of ∞-categories, which provides a more convenient formalism for our purposes.

References

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