Cohomological Formulation (Lecture 3)

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Let $\mathbf{F}_q$ be a finite field with q elements, let X be an algebraic curve over $\mathbf{F}_q$, and let G be a smooth affine group scheme over X with connected fibers. Let $d = \dim(G)$. For each closed point $x \in X$, let $\deg(x)$ denote the degree of x (that is, the dimension of the residue field $\kappa(x)$ as a vector space over $\mathbf{F}_q$) and let $\mathcal{L}$ denote the line bundle on X of left-invariant top forms on G. In the previous lecture, we formulated the function field analogue of Weil's conjecture:

Conjecture 1 (Weil)**.** Suppose that the generic fiber of G is semisimple and simply connected. Then

$$\frac{\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}}{q^{d(g-1)+\deg(\mathcal{L})}} = \prod_{x \in X} \frac{q^{d \deg(x)}}{|G(\kappa(x))|}.$$

Remark 2. Note that neither side of the equation of Conjecture 1 is *a priori* well defined. The absolute convergence of the product on the right hand side is equivalent to the well-definedness of Tamagawa measure. The left hand side is usually an infinite sum (unless the generic fiber of G is anisotropic), but the conjecture asserts that this infinite sum converges to the right hand side.

The assertion of Conjecture 1 can be regarded as a function field analogue of the Siegel mass formula (in its original formulation). However, there are tools available for attacking Conjecture 1 that have no analogue in the case of a number field. More specifically, we would like to take advantage of the fact that the collection of all G-bundles on X admits an algebro-geometric parametrization.

Definition 3. If Y is a scheme equipped with a map $Y \to X$, we define a *G-bundle* on Y to be a principal homogeneous space for the group scheme $G_Y = Y \times_X G$ over Y. The collection of G-bundles on Y forms a category (in which all morphisms are isomorphisms).

For every $\mathbf{F}_q$-algebra R, we let $\operatorname{Bun}_G(X)$ denote the category of G-bundles on the relative curve

$$\operatorname{Spec} R \times_{\operatorname{Spec} \mathbf{F}_q} X.$$

The construction $R \mapsto \operatorname{Bun}_G(X)(R)$ is an example of an *algebraic stack*, which we will denote by $\operatorname{Bun}_G(X)$. We will refer to $\operatorname{Bun}_G(X)$ as the *moduli stack of G-bundles on X*.

Remark 4. The algebraic stack $\operatorname{Bun}_G(X)$ is smooth over $\mathbf{F}_q$, and its dimension is given by $d(g-1)+\deg(\mathcal{L})$. By definition, the category $\operatorname{Bun}_G(X)(\mathbf{F}_q)$ is the category of G-bundles on X. We will denote the sum $\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}$ by $|\operatorname{Bun}_G(X)(\mathbf{F}_q)|$: we can think of it as a (weighted) count of the objects of $\operatorname{Bun}_G(X)(\mathbf{F}_q)$, which properly takes into account the fact that $\operatorname{Bun}_G(X)(\mathbf{F}_q)$ is a category rather than a set. With this notation, we can rephrase Conjecture 1 as an equality

$$\frac{|\operatorname{Bun}_G(X)(\mathbf{F}_q)|}{q^{\dim(\operatorname{Bun}_G)}} = \prod_{x \in X} \frac{q^{d \deg(x)}}{|G(\kappa(x))|}$$

Variant 5. For each closed point $x \in X$, let $\mathrm{Bun}_G(x)$ denote the stack whose R-valued points are principal G-bundles on the X-scheme $\mathrm{Spec}(R \otimes_{\mathbf{F}_q} \kappa(x))$. Then $\mathrm{Bun}_G(x)$ is also a smooth algebraic stack (it is the classifying stack of the group scheme over $\mathbf{F}_q$ obtained by Weil restriction of G along the morphism $\mathrm{Spec}\,\kappa(x) \to \mathrm{Spec}\,\mathbf{F}_q$) having dimension $-d\deg(x)$. Since the fibers of G are connected, every G-bundle on $\mathrm{Spec}\,\kappa(x)$ is trivial (by virtue of Lang's theorem), and the trivial G-bundle on $\mathrm{Spec}\,\kappa(x)$ has automorphism group $G(\kappa(x))$. We may therefore write

$$\frac{q^{d\deg(x)}}{|G(\kappa(x))|} = \frac{|\mathrm{Bun}_G(x)(\mathbf{F}_q)|}{q^{\dim \mathrm{Bun}_G(x)}}.$$

Conjecture 1 can therefore be written in the more suggestive form

$$\frac{|\mathrm{Bun}_G(X)(\mathbf{F}_q)|}{q^{\dim \mathrm{Bun}_G(X)}} = \prod_{x \in X} \frac{|\mathrm{Bun}_G(x)(\mathbf{F}_q)|}{q^{\dim \mathrm{Bun}_G(x)}}.$$

Heuristically, this formula reflects the idea that $\mathrm{Bun}_G(X)$ can be obtained as a "continuous product" of the algebraic stacks $\{\mathrm{Bun}_G(x)\}_{x \in X}$.

The problem of counting the number of points on algebraic varieties over $\mathbf{F}_q$ is the subject of another very famous idea of Weil. Let Y be a quasi-projective variety over $\mathbf{F}_q$, so that there exists an embedding $j : Y \hookrightarrow \mathbf{P}^n_{\mathbf{F}_q}$ for some $n \geq 0$. Set $\overline{Y} = \mathrm{Spec}\,\overline{\mathbf{F}}_q \times_{\mathrm{Spec}\,\mathbf{F}_q} Y$, so that j determines an embedding $\overline{j} : \overline{Y} \to \mathbf{P}^n_{\overline{\mathbf{F}}_q}$. There is a canonical map from $\mathbf{P}^n_{\overline{\mathbf{F}}_q}$ to itself, given in homogeneous coordinates by

$$[x_0 : \cdots : x_n] \mapsto [x_0^q : \cdots x_n^q].$$

Since $\overline{Y}$ is characterized by the vanishing and nonvanishing of homogeneous polynomials with coefficients in $\mathbf{F}_q$, this construction determines a map $\varphi : \overline{Y} \to \overline{Y}$, which we will refer to as the *geometric Frobenius map*. Then the finite set $Y(\mathbf{F}_q)$ of $\mathbf{F}_q$-points of Y can be identified with the *fixed locus* of the map $\varphi : \overline{Y} \to \overline{Y}$. Weil made a series of conjectures about the behavior of the integers $|Y(\mathbf{F}_q)|$, which are informed by the following heuristic:

Idea 6 (Weil)**.** There exists a good cohomology theory for algebraic varieties having the property that the fixed point set $Y(\mathbf{F}_q) = \{y \in \overline{Y}(\overline{\mathbf{F}}_q) : \varphi(y) = y\}$ is given by the Lefschetz fixed point formula

$$Y(\mathbf{F}_q) = \sum_{i \geq 0} (-1)^i \,\mathrm{Tr}(\varphi |\, \mathrm{H}^i_c(\overline{Y})).$$

Weil's conjectures were eventually proven by Grothendieck and Deligne by introducing the theory of *ℓ-adic cohomology*, which we will denote by $\overline{Y} \mapsto \mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)$ (there is also an analogue of ℓ-adic cohomology with compact supports which appears in the fixed point formula above, which we will denote by $\mathrm{H}^*_c(\overline{Y}; \mathbf{Q}_\ell)$). Here ℓ denotes some fixed prime number which does not divide q.

For our purposes, it will be convenient to write the Grothendieck-Lefschetz trace formula in a slightly different form. Suppose that Y is a smooth variety of dimension n over $\mathbf{F}_q$. Then, from the perspective of ℓ-adic cohomology, Y behaves as if it were a smooth manifold of dimension $2n$. In particular, it satisfies Poincare duality: that is, there is a perfect pairing

$$\mu : \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) \to \mathbf{Q}_\ell\,.$$

This pairing is not quite φ-equivariant: instead, it fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) & \xrightarrow{\mu} & \mathbf{Q}_\ell \\ \downarrow{\scriptstyle \varphi \otimes \varphi} & & \downarrow{\scriptstyle q^n} \\ \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) & \xrightarrow{\mu} & \mathbf{Q}_\ell\,. \end{array}$$

It follows formally that

$$q^{-n}\operatorname{Tr}(\varphi|\operatorname{H}^i_c(\overline{Y};\mathbf{Q}_\ell)) \simeq \operatorname{Tr}(\varphi^{-1}|\operatorname{H}^{2n-i}(\overline{Y};\mathbf{Q}_\ell)).$$

We may therefore rewrite the Grothendieck-Lefschetz trace formula as an equality

$$\sum_{i\geq 0}(-1)^i \operatorname{Tr}(\varphi^{-1}|\operatorname{H}^i(\overline{Y};\mathbf{Q}_\ell)) = \frac{|Y(\mathbf{F}_q)|}{q^{\dim(Y)}}.$$

This suggests the possibility of breaking Conjecture 1 into two parts.

Notation 7. In what follows, we fix an embedding $\mathbf{Q}_\ell \hookrightarrow \mathbf{C}$, so that we can regard the trace of any endomorphism of a $\mathbf{Q}_\ell$-vector space as a complex number. Let $\overline{\operatorname{Bun}}_G(X)$ denote the fiber product $\operatorname{Spec}\overline{\mathbf{F}}_q \times_{\operatorname{Spec}\mathbf{F}_q} \operatorname{Bun}_G(X)$, so that $\overline{\operatorname{Bun}}_G(X)$ is a smooth algebraic stack over $\overline{\mathbf{F}}_q$ equipped with a geometric Frobenius map $\varphi : \overline{\operatorname{Bun}}_G(X) \to \overline{\operatorname{Bun}}_G(X)$.

Conjecture 8. The algebraic stack $\operatorname{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula

$$\frac{|\operatorname{Bun}_G(X)(\mathbf{F}_q)|}{q^{\dim(\operatorname{Bun}_G(X))}} = \sum_{i\geq 0}(-1)^i \operatorname{Tr}(\varphi^{-1}|\operatorname{H}^i(\overline{\operatorname{Bun}}_G(X);\mathbf{Q}_\ell)).$$

Conjecture 9. There is an equality

$$\sum_{i\geq 0}(-1)^i \operatorname{Tr}(\varphi^{-1}|\operatorname{H}^i(\overline{\operatorname{Bun}}_G(X);\mathbf{Q}_\ell)) = \prod_{x\in X}\frac{q^{d\deg(x)}}{|G(\kappa(x))|}.$$

Warning 10. The ℓ-adic cohomology ring $\operatorname{H}^*(\overline{\operatorname{Bun}}_G(X);\mathbf{Q}_\ell)$ is generally nonzero in infinitely many degrees, so that the sum

$$\sum_{i\geq 0}(-1)^i \operatorname{Tr}(\varphi^{-1}|\operatorname{H}^i(\overline{\operatorname{Bun}}_G(X);\mathbf{Q}_\ell))$$

appearing in Conjectures 8 and 9 is generally infinite. Consequently, the convergence of this sum is not automatic, but part of the conjecture. Similarly, neither of the expressions

$$\frac{|\operatorname{Bun}_G(\mathbf{F}_q)|}{q^{\dim\operatorname{Bun}_G(X)}} = \frac{\sum_{\mathcal{P}}\frac{1}{|\operatorname{Aut}(\mathcal{P})|}}{q^{\dim\operatorname{Bun}_G(X)}} \qquad \prod_{x\in X}\frac{q^{d\deg(x)}}{|G(\kappa(x))|}$$

is obviously convergent.

The proofs of Conjecture 8 and 9 require completely different techniques. The remainder of this course will be devoted to the proof of Conjecture 9. Roughly speaking, the idea is to use the heuristic that $\operatorname{Bun}_G(X)$ is a "continuous product" of the classifying stacks $\operatorname{Bun}_G(x)$ as x ranges over the points of X to formulate a local-to-global principle which can be used to compute the cohomology $\operatorname{H}^*(\overline{\operatorname{Bun}}_G(X);\mathbf{Q}_\ell)$ in terms of "local" data. This strategy is inspired by similar principles in a purely topological setting, which we will discuss in the next lecture.

For the remainder of this lecture, we will briefly sketch the work that needs to be done to prove Conjecture 8, following the work of Kai Behrend. The first problem that we need to wrestle with is that $\operatorname{Bun}_G(X)$ is an algebraic stack over $\mathbf{F}_q$, rather than an algebraic variety over $\mathbf{F}_q$. This is because G-bundles over the curve X admit nontrivial automorphisms. However, we can rigidify the situation by considering *G-bundles with level structure*.

Fix a closed point $x \in X$. Let $\mathcal{O}_x$ be the completed local ring of x in X and let $\mathfrak{m}_x$ denote its maximal ideal. For each $N \geq 0$, we let D_N denote the affine scheme $\operatorname{Spec}\mathcal{O}_x/\mathfrak{m}_x^N$, which we regard as a closed subscheme of X (the closed subscheme determined by the effective divisor Nx).

For every $\mathbf{F}_q$-algebra R, let $\operatorname{Bun}_G^{(N)}(X)(R)$ denote the category whose objects are G-bundles on the relative curve $\operatorname{Spec} R \times_{\operatorname{Spec}\mathbf{F}_q} X$ equipped with a trivialization on the closed subscheme $\operatorname{Spec} R \times_{\operatorname{Spec}\mathbf{F}_q} D_N$.

One can show that the construction $R \mapsto \mathrm{Bun}_G^{(N)}(X)(R)$ determines an algebraic stack $\mathrm{Bun}_G^{(N)}(X)$, equipped with a map $\mathrm{Bun}_G^{(N)}(X) \to \mathrm{Bun}_G(X)$. Moreover, this map exhibits $\mathrm{Bun}_G^{(N)}(X) \to \mathrm{Bun}_G(X)$ as a principal H-bundle over $\mathrm{Bun}_G(X)$, where H denotes the Weil restriction of the group $D_N \times_X G$ along the map $D_N \to \operatorname{Spec} \mathbf{F}_q$ (so that H is a smooth connected algebraic group of dimension $dN \deg(x)$ over $\mathbf{F}_q$).

Let $\mathcal{P}$ be a G-bundle on X. Then $\mathcal{P}$ might admit nontrivial automorphisms. However, any nontrivial automorphism α of $\mathcal{P}$ must act nontrivially on $\mathcal{P}|_{D_N}$ provided that N is sufficiently large. Moreover, this result is true in families. More precisely, suppose that we are given a *quasi-compact* $\mathbf{F}_q$-scheme V equipped with a map $V \to \mathrm{Bun}_G(X)$, classifying a G-bundle $\mathcal{P}$ on $V \times_{\operatorname{Spec} \mathbf{F}_q} X$. Then automorphisms of $\mathcal{P}$ are classified by an affine group scheme $\mathrm{Aut}(\mathcal{P})$ over V. Moreover, for $N \gg 0$, the group scheme $\mathrm{Aut}(\mathcal{P})$ acts faithfully on the restriction of $\mathcal{P}$ to $V \times_{\operatorname{Spec} \mathbf{F}_q} D_N$.

Suppose that we are given an open substack $U \subseteq \mathrm{Bun}_G(X)$ which is *quasi-compact*. It follows from the above reasoning that we can choose $N \gg 0$ such that the fiber product $U^{(N)} = U \times_{\mathrm{Bun}_G(X)} \mathrm{Bun}_G^{(N)}(X)$ is a set-valued functor, rather than a category-valued functor. It is therefore representable by a smooth algebraic space over $\mathbf{F}_q$ (which is quasi-compact and quasi-separated).

Since the map $G \to X$ is smooth with connected fibers, every G-torsor $\mathcal{P}$ on X is trivial when restricted to the divisor D_N. Consequently, we can identify the set of isomorphism classes of objects of the category $U(\mathbf{F}_q)$ with the quotient of the set $U^{(N)}(\mathbf{F}_q)$ by the action of the finite group $H(\mathbf{F}_q)$. Moreover, the automorphism group of an object $\mathcal{P} \in U(\mathbf{F}_q)$ can be identified with the stabilizer in $H(\mathbf{F}_q)$ of an arbitrary lift of $\mathcal{P}$ to $U^{(N)}(\mathbf{F}_q)$. We therefore have an equality

$$|U(\mathbf{F}_q)| = \sum_{\mathcal{P} \in U(\mathbf{F}_q)} \frac{1}{|\operatorname{Aut}(\mathcal{P})|} = \frac{|U^{(N)}(\mathbf{F}_q)|}{|H(\mathbf{F}_q)|}.$$

Here the numerator and denominator can be computed by means of the classical Grothendieck-Lefschetz fixed point theorem:

$$|U^{(N)}(\mathbf{F}_q)| = q^{\dim(U^{(N)})} \sum_{i \geq 0} (-1)^i \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^i(\overline{U}^{(N)}; \mathbf{Q}_\ell))$$

$$|H(\mathbf{F}_q)| = q^{\dim(H)} \sum_{i \geq 0} (-1)^i \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^i(H; \mathbf{Q}_\ell)).$$

The algebraic stack U can be described as the stack-theoretic quotient of $U^{(N)}$ by the action of H. Consequently, the ℓ-adic cohomology of $\overline{U}$ can be computed by a cobar spectral sequence with first page given by

$$E_1^{*,t} = \mathrm{H}^*(\overline{U}^{(N)}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^*_{\mathrm{red}}(\overline{H}; \mathbf{Q}_\ell)^{\otimes t}.$$

Modulo issues of convergence, we may therefore compute

$$\begin{aligned}
\sum_i (-1)^i \operatorname{Tr}(\varphi^{-1} | \operatorname{H}^i(\overline{U}; \mathbf{Q}_\ell)) &= \sum_{s,t} (-1)^{s+t} \operatorname{Tr}(\varphi^{-1} | E_\infty^{s,t}) \\
&= \sum_{s,t} (-1)^{s+t} \operatorname{Tr}(\varphi^{-1} | E_1^{s,t}) \\
&= \sum_t (-1)^t \operatorname{Tr}(\varphi^{-1} | \operatorname{H}^*(\overline{U}^{(N)}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \operatorname{H}^*_{\mathrm{red}}(\overline{H}; \mathbf{Q}_\ell)^{\otimes t}) \\
&= \sum_t (-1)^t \operatorname{Tr}(\varphi^{-1} | \operatorname{H}^*(\overline{U}^{(N)}; \mathbf{Q}_\ell)) \operatorname{Tr}(\varphi^{-1} | \operatorname{H}^*_{\mathrm{red}}(\overline{H}; \mathbf{Q}_\ell))^t \\
&= \sum_t (-1)^t \frac{|U^{(N)}(\mathbf{F}_q)|}{q^{\dim(U^{(N)})}} \left(\frac{|H(\mathbf{F}_q)|}{q^{\dim(H)}} - 1\right)^t \\
&= \frac{|U^{(N)}(\mathbf{F}_q)|}{q^{\dim(U^{(N)})}} \frac{q^{\dim(H)}}{|H(\mathbf{F}_q)|} \\
&= \frac{1}{q^{\dim(U)}} \frac{|U^{(N)}(\mathbf{F}_q)|}{|H(\mathbf{F}_q)|} \\
&= \frac{|U(\mathbf{F}_q)|}{q^{\dim(U)}}.
\end{aligned}$$

Using the connectedness of H, one can see that all of these sums are absolutely convergent; for more details, we refer the reader to [2].

Remark 11. The convergence above relies crucially on the fact that we are working with eigenvalues of the arithmetic Frobenius φ^{-1} (which are small), rather than the eigenvalues of the geometric Frobenius φ (which are large). This is our reason for preferring the variant formulation of the Grothendieck-Lefschetz fixed point theorem.

If the algebraic stack $\operatorname{Bun}_G(X)$ were quasi-compact, we could take $U = \operatorname{Bun}_G(X)$ in the preceding considerations, and thereby obtain a proof of Conjecture 8. Unfortunately, the algebraic stack $\operatorname{Bun}_G(X)$ is *never* quasi-compact, except in the case where the group G is trivial. Consequently, we need to work much harder.

Suppose that we are given a stratification of $\operatorname{Bun}_G(X)$ where each stratum admits a finite radicial morphism from a smooth stack U_α with a map $U_\alpha \to \operatorname{Bun}_G(X)$ of relative dimension $-d_\alpha$. Applying the above argument to each U_α individually, we obtain

$$\begin{aligned}
\frac{|\operatorname{Bun}_G(X)(\mathbf{F}_q)|}{q^{\dim(\operatorname{Bun}_G(X))}} &= \sum_\alpha \frac{|U_\alpha(\mathbf{F}_q)|}{q^{\dim(U_\alpha)+d_\alpha}} \\
&= \sum_\alpha \frac{1}{q^{d_\alpha}} \sum_i (-1)^{-1} \operatorname{Tr}(\varphi^{-1} | \operatorname{H}^i(\overline{U}_\alpha; \mathbf{Q}_\ell)) \\
&= \sum_\alpha \sum_i (-1)^{-1} \operatorname{Tr}(\varphi^{-1} | \operatorname{H}^i(\overline{U}_\alpha; \mathbf{Q}_\ell(-d_\alpha)).
\end{aligned}$$

The cohomology groups $\operatorname{H}^*(\overline{\operatorname{Bun}}_G; \mathbf{Q}_\ell)$ and $\{\operatorname{H}^*(\overline{U}_\alpha; \mathbf{Q}_\ell(-d_\alpha))\}$ are related by a spectral sequence. Modulo convergence issues, this yields a proof of Conjecture 8. However, to guarantee convergence, one needs to choose the U_α carefully. In the case where the group scheme G is split and reductive, one can employ the Harder-Narasimhan stratification (see [1] for details). To handle the general case, we need a generalization of the Harder-Narasimhan stratification to the case where the group scheme G is non-constant, and may have bad reduction at finitely many points of X. We will discuss this generalization elsewhere.

Example 12. Let $G = \mathrm{SL}_2$, so that $\mathrm{Bun}_G(X)$ is the moduli stack of rank 2 vector bundles $\mathcal{E}$ over X equipped with a trivialization of the determinant bundle $\det(\mathcal{E})$. Recall that a G-bundle $\mathcal{E}$ is said to be *semistable* if there does not exist a line subbundle $\mathcal{L} \subseteq \mathcal{E}$ of positive degree. One can show that semistable G-bundles are classified by a quasi-compact open substack $\mathrm{Bun}_G^{\mathrm{ss}}(X) \subseteq \mathrm{Bun}_G(X)$.

If a G-bundle $\mathcal{E}$ is not semistable, then (by definition) there exists a line subbundle $\mathcal{L} \subseteq \mathcal{E}$ of positive degree. In this case, the line bundle $\mathcal{L}$ is uniquely determined. The collection of those G-bundles $\mathcal{E}$ for which $\deg(\mathcal{L}) = d > 0$ are classified by a locally closed substack $\mathrm{Bun}_G^d(X) \subseteq \mathrm{Bun}_G(X)$. Note that if $\mathcal{E}$ is such a bundle, then the data of a trivialization of $\det(\mathcal{E})$ is equivalent to the data of an isomorphism $\mathcal{E} / \mathcal{L} \simeq \mathcal{L}^{-1}$, giving us an exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1} \to 0.$$

Consequently, $\mathrm{Bun}_G^d(X)$ can be identified (at least at the level of field-valued points) with the moduli stack classifying pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is a line bundle of degree d and α is an extension of $\mathcal{L}^{-1}$ by $\mathcal{L}$. If X has an $\mathbf{F}_q$-rational point, then each of these moduli stacks has the ℓ-adic cohomology of the moduli stack $\mathrm{Bun}_{\mathbf{G}_m}^0(X)$ of degree 0 line bundles on X. However, the dimension $\dim(\mathrm{Bun}_G^d(X)) = 2g - 2 - 2d$ approaches $-\infty$ as $d \to \infty$.

References

[1] Behrend, K. *The Lefschetz Trace Formula for the Moduli Stack of Principal Bundles.* PhD dissertation.

[2] Behrend, K. *Derived ℓ-adic categories for algebraic stacks.* Memoirs of the American Mathematical Society Vol. 163, 2003.