Recall that finite polyhedra $X$ and $Y$ are concordant if there is a piecewise-linear fibration $q : E \to [0,1]$ with $X \simeq q^{-1}\{0\}$ and $Y \simeq q^{-1}\{1\}$. In the last lecture, we asserted that $X$ and $Y$ are simply homotopy equivalent if and only if they are concordant, and proved the "if" direction. Our goal in this lecture is to use this fact as a starting point for the study of “higher” simple homotopy theory, following ideas of Hatcher.

For any finite polyhedron $B$, we can contemplate piecewise-linear fibrations $q : E \to B$ (where $E$ is also a finite polyhedron). Our first goal is to construct a universal example of such a fibration, so that the base $B$ can be regarded as a classifying space for PL fibrations. It is not clear that such a classifying space exists in the setting of finite polyhedra, but we can give an almost tautological construction of one in the setting of simplicial sets.

**Definition 1.** For each integer $n$, let $\Delta^n$ denote the topological simplex of dimension $n$ and let $M_n$ denote the set of all finite polyhedra $E \subseteq \Delta^n \times \mathbb{R}^\infty$ for which the projection map $E \to \Delta^n$ is a fibration.

Note that for any linear map of simplices $\alpha : \Delta^m \to \Delta^n$, the construction $E \mapsto E \times \Delta^m \Delta^n$ defines a map of sets $\alpha^* : M_n \to M_m$. In particular, we can regard the construction $[n] \mapsto M_n$ as a simplicial set, which we will denote by $M$.

Before we analyze the simplicial set $M$, we need a few general facts about the relationship between polyhedra and simplicial sets.

**Remark 2.** Let $K_0$, $K_1$, and $K_{01}$ be polyhedra, and suppose we are given piecewise linear embeddings

$$K_0 \xleftarrow{i_0} K_{01} \xrightarrow{i_1} K_1.$$  

Then the pushout $K_0 \amalg_{K_{01}} K_1$ exists in the category of polyhedra: that is, we can regard endow $K_0 \amalg_{K_{01}} K_1$ with the structure of a polyhedron, where a map $K_0 \amalg_{K_{01}} K_1 \to L$ is piecewise linear if and only if its restriction to $K_0$ and $K_1$ is piecewise linear.

Beware that this need not be true if $i_0$ is not an embedding, even if $i_1$ is an embedding. This is often a technical nuisance.

**Example 3.** Let $X$ be a finite simplicial set. We say that $X$ is nonsingular if every simplex $\sigma : \Delta^n \to X$ is either degenerate (meaning that it factors through $\Delta^m$ for $m < n$) or is a monomorphism of simplicial sets (in particular, all the faces of $\sigma$ are again nondegenerate).

For any nonsingular finite simplicial set $X$, the geometric realization $|X|$ can be regarded as a finite polyhedron. More precisely, there is a unique PL structure on $|X|$ having the property that for every nondegenerate $n$-simplex of $X$, the associated map $\Delta^n \to |X|$ is piecewise linear (this follows by invoking Remark 2 repeatedly).

In what follows, we will often not distinguish between a (finite nonsingular) simplicial set $X$ and the polyhedron $|X|$. For example, we use the symbol $\Delta^n$ to denote both the $n$-simplex as a simplicial set and the topological $n$-simplex, and apply similar considerations to the boundary $\partial \Delta^n$ and the horns $\Lambda^n_i \subseteq \Delta^n$.

We will also need the following technical fact, whose proof we omit (see Lemma 2.7.12 of [1]):

$$\text{...}$$
Proposition 4. Let $q : E \to B$ be a map of finite polyhedra. The following conditions are equivalent:

1. The map $q$ is a fibration.

2. For every triangulation of $B$ and every simplex $\sigma$ of the triangulation, the induced map $E \times_B \sigma \to \sigma$ is a fibration.

3. There exists a triangulation of $B$ such that, for every simplex $\sigma$ of the triangulation, the induced map $E \times_B \sigma \to \sigma$ is a fibration.

Corollary 5. Let $B$ be a finite nonsingular simplicial set. Then $\text{Hom}(B, \mathcal{M})$ can be identified with the set of finite polyhedra $E \subseteq |B| \times \mathbb{R}^\infty$ for which the projection map $E \to |B|$ is a fibration.

Proof. The geometric realization $|B|$ admits a triangulation for which each simplex is contained in the image of some simplex of $B$ (beware that the nondegenerate simplices of $B$ do not generally themselves determine a triangulation of $|B|$, unless one is liberal with the meaning of the word “triangulation”).

Corollary 6. The simplicial set $\mathcal{M}$ is a Kan complex.

Proof. Suppose we are given a map $f_0 : \Lambda^n_i \to \mathcal{M}$, given by a polyhedron $E \subseteq |\Lambda^n_i| \times \mathbb{R}^\infty$ for which the projection $E \to |\Lambda^n_i|$ is a fibration. Choose a piecewise linear retraction $r : |\Delta^n| \to |\Lambda^n_i|$, and define $E' = E \times_{|\Lambda^n_i|} |\Delta^n|$. Then $E'$ can be identified with a map $f : \Delta^n \to \mathcal{M}$ extending $f_0$.

We next investigate the role of the Kan complex $\mathcal{M}$ as a “classifying space.”

Exercise 7. Let $B$ be a finite polyhedron. Suppose we are given fibrations of finite polyhedra $f : X \to B$, $g : Y \to B$. We will say that $f$ and $g$ are concordant if there exists a fibration of finite polyhedra $h : Z \to B \times [0,1]$ for which the inverse image of $B \times \{0\}$ is isomorphic to $X$ and the inverse image of $B \times \{1\}$ is isomorphic to $Y$. Show that concordance is an equivalence relation.

Let $B$ be a finite nonsingular simplicial set. Any map $f : B \to \mathcal{M}$ determines a fibration of finite polyhedra $E_f \to |B|$, and any homotopy between maps $f, g : |B| \to \mathcal{M}$ determines a concordance from $E_f$ to $E_g$. We therefore obtain a well-defined map from the set $[B, \mathcal{M}]$ of homotopy classes of maps from $B$ into $\mathcal{M}$ to the set of concordance classes of fibrations over $|B|$.

Proposition 8. This map is bijective.

Proof. To prove surjectivity, it suffices to note that for any map of finite polyhedra $X \to |B|$, we can choose a compatible PL embedding of $X$ into $|B| \times \mathbb{R}^\infty$.

To prove injectivity, it suffices to show that if $X \subseteq |B| \times \mathbb{R}^\infty$ and $Y \subseteq |B| \times \mathbb{R}^\infty$ are polyhedra fibered over $|B|$ and we are given any concordance $Z \to |B \times \Delta^1|$ from $X$ to $Y$, then we can choose a PL embedding of $Z$ into $|B \times \Delta^1| \times \mathbb{R}^\infty$ which is compatible with the given embeddings on $X$ and $Y$.

References