The Setup (Lecture 35)

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Let us begin by recalling some of our main characters.

**Notation 1.** We let $M$ denote the “classifying space” for simple homotopy types: that is, the Kan complex whose $k$-simplices are finite polyhedra $E \subseteq \Delta^k \times \mathbb{R}^\infty$ for which the projection map $E \to \Delta^k$ is a fibration.

For each integer $d \geq 0$, we let $\text{Man}^d$ denote the Kan complex whose $k$-simplices are pairs $(E, \rho)$, where $E \subseteq \Delta^k \times \mathbb{R}^\infty$ is a finite polyhedron, the projection $E \to \Delta^k$ is a PL fiber bundle whose fibers are PL manifolds of dimension $d$ (possibly with boundary), and $\rho$ is a trivialization of the (relative) tangent microbundle $T_{E/\Delta^k}$.

Note that the construction $(E, \rho) \mapsto E$ determines a map of simplicial sets $\theta_d : \text{Man}^d \to M$.

Forming the product with $[0, 1]$ (and composing with an embedding $\mathbb{R}^\infty \times [0, 1] \hookrightarrow \mathbb{R}^\infty$) we obtain stabilization maps

$$M \to M \to M \to \ldots$$

$$\text{Man}^0 \to \text{Man}^1 \to \text{Man}^2 \to \ldots$$

which are compatible with the forgetful maps $\theta_d$. We therefore obtain a map of simplicial sets

$$\theta_\infty : \text{Man}^\infty = \lim_{\to \delta} \text{Man}^d \to \lim_{\to \delta} M \cong M.$$

**Variant 2.** For each integer $d \geq 0$, we define simplicial sets $A^d, B^d, C^d,$ and $D^d$ as follows:

(a) A $k$-simplex of $A^d$ is a $k$-simplex $(E, \rho)$ of $\text{Man}^d$ having the property that each component of each fiber of $E$ has nonempty boundary.

(b) A $k$-simplex of $A^d$ is a pair $(E, \rho)$ where $E \subseteq \Delta^k \times \mathbb{R}^\infty$ is a finite polyhedron, the projection $E \to \Delta^k$ is a fiber bundle whose fibers are PL manifolds of dimension $d$, each component of which has nonempty boundary, and $\rho$ is a PL immersion from $E$ to $\Delta^k \times \mathbb{R}^d$ which commutes with the projection to $\Delta^k$.

(c) A $k$-simplex of $C^d$ is a $k$-simplex $(E, \rho)$ of $B^d$ where the map $\rho$ is an embedding.

(d) A $k$-simplex of $D^d$ is a finite polyhedron $E \subseteq \Delta^k \times \mathbb{R}^d$ for which the projection $E \to \Delta^k$ is a fiber bundle whose fibers are PL manifolds of dimension $d$.

Note that every immersion of a PL $d$-manifold into $\mathbb{R}^d$ determines a trivialization of its tangent microbundle.

We therefore have canonical maps of simplicial sets

$$D^d \leftarrow C^d \subseteq B^d \to A^d \subseteq \text{Man}^d.$$

Moreover, there are evident stabilization maps for each of these simplicial sets (which increase $d$ by 1), given by forming the product with $[0, 1]$. We make the following observations:

- The map $\lim_{\to \delta} A^d \to \lim_{\to \delta} \text{Man}^d$ is an isomorphism of simplicial sets. This follows from the observation that each of the stabilization maps $\text{Man}^d \to \text{Man}^{d+1}$ factors through $A^{d+1} \subseteq \text{Man}^{d+1}$ (since a product $M \times [0, 1]$ always has nonempty boundary).
• Each of the maps $B^d \to A^d$ is a homotopy equivalence of Kan complexes. This follows from the work of Haefliger-Poenaru on piecewise linear immersion theory ([1]).

• Each of the maps $C^d \to D^d$ is a trivial Kan fibration (this is trivial: it essentially amounts to the observation that the space of embeddings from a PL manifold $M$ into $\mathbb{R}^\infty$ is contractible).

• The inclusion $\lim D^d \to \lim C^d$ is a homotopy equivalence. This follows from elementary general position arguments. For example, suppose that we wish to prove surjectivity on $\pi_0$. Unwinding the definitions, we wish to show that if $M$ is a PL $d$-manifold equipped with an immersion $\rho : M \to \mathbb{R}^d$, then after replacing $M$ by some product $M \times [0, 1]^k$ we can arrange that $\rho$ is isotopic (through immersions) to an embedding. To prove this, we choose $k \gg 0$ and an embedding $e = (e_1, \ldots, e_k) : M \to [0, 1]^k$.

Using the fact that $\rho$ is an immersion, we deduce that there exists $\epsilon > 0$ for which the map

$$M \times [0, 1]^k \to \mathbb{R}^{d+k}$$

$$(x, t_1, \ldots, t_k) \mapsto (\rho(x), e_1(x) + \epsilon t_1, \ldots, e_k(x) + \epsilon t_k)$$

is an embedding, and it is not difficult to check that this embedding is PL isotopic (through immersions) to the $\rho \times \text{id}$.

It follows that we obtain homotopy equivalences

$$\lim D^d \leftarrow \lim C^d \subseteq \lim B^d \to \lim A^d \subseteq \lim \text{Man}^d.$$ 

In other words, the direct limit $\text{Man}^\infty = \lim \text{Man}^d$ can be identified with a classifying space for embedded PL submanifolds $M \subseteq \mathbb{R}^d$ (stabilized by taking the dimension $d$ to infinity).

Our goal over the next several lectures is to prove the following:

**Theorem 3.** The map $\theta_\infty$ is a homotopy equivalence of Kan complexes.

Let us begin by trying to analyze an individual map $\theta_d$. By construction $\text{Man}^d$ is a classifying space for compact framed PL manifolds of dimension $d$. Consequently, $\text{Man}^d$ is homotopy equivalent to a disjoint union

$$\Pi_M \text{BAut}(M)$$

where the disjoint union is taken over all isomorphism classes of compact framed PL manifolds $M$ of dimension $d$, and $\text{Aut}(M)$ denotes the (simplicial) group of framed PL homeomorphisms of $M$ with itself. Let us make this identification more explicit. In what follows, we will use the term “simplicial category” to mean a simplicial object in the category of categories, and we will use the term “simplicially enriched category” to mean a category enriched over simplicial sets: that is, a simplicial category where the simplicial set of objects is constant.

**Notation 4.** Fix an integer $d \geq 0$. For each $k \geq 0$, we let $\mathcal{E}_k$ denote the category whose objects are pairs $(E, \rho)$, where $E \subseteq \Delta^k \times \mathbb{R}^\infty$ is a finite polyhedron, the projection $E \to \Delta^k$ is a PL fiber bundle whose fibers are PL manifolds of dimension $d$ (possibly with boundary), and $\rho$ is a trivialization of the (relative) tangent microbundle $T_{E/\Delta^k}$. A morphism from $(E, \rho)$ to $(E', \rho')$ is a PL homeomorphism of $E$ with $E'$, compatible with the projection to $\Delta^k$, which carries $\rho$ to $\rho'$. We will regard $\mathcal{E}_\bullet$ as a simplicial category. Let $\mathcal{E}_0$ denote the “underlying” simplicially enriched category, whose objects are the objects of the category $\mathcal{E}_0$ (which we can identify with framed PL $d$-manifolds, if we ignore the data of an embedding into $\mathbb{R}^\infty$).

It is not difficult to see that the homotopy type of the disjoint union $\Pi_M \text{BAut}(M)$ is modeled by the bisimplicial set $N_{\bullet}(\mathcal{E}_\bullet)$. On the other hand, we have a canonical isomorphism of simplicial sets $\text{Man}^d \simeq N_0(\mathcal{E}_\bullet)$. The existence of a homotopy equivalence $\text{Man}^d \simeq \Pi_M \text{BAut}(M)$ is a consequence of the following:
Proposition 5. The canonical maps

\[ N_\bullet (\mathcal{C}_\bullet ^o) \hookrightarrow N_\bullet (\mathcal{C}_\bullet ) \leftarrow N_0(\mathcal{C}_\bullet ) = \text{Man}^d \]

are weak homotopy equivalences (of bisimplicial sets).

Proof. The first map is a weak homotopy equivalence because for each integer \( k \), the inclusion \( \mathcal{C}_k ^o \hookrightarrow \mathcal{C}_k \) is an equivalence of categories (this follows from the observation that any PL fiber bundle \( E \to \Delta^k \) is trivial, because \( \Delta^k \) is contractible). To show that the second map is a weak homotopy equivalence, it will suffice to show that for each integer \( n \), the degeneracy map

\[ N_0(\mathcal{C}_\bullet ) \hookrightarrow N_n(\mathcal{C}_\bullet ) \]

is a homotopy equivalence of Kan complexes. This map has a left inverse \( q \), given by evaluation at any choice of vertex in \( \Delta^n \). It now suffices to show that the map \( q : N_n(\mathcal{C}_\bullet ) \to N_0(\mathcal{C}_\bullet ) \) is a trivial Kan fibration. This follows from the contractibility of the space of embeddings of a PL manifold \( M \) into \( \mathbb{R}^\infty \); we leave the details to the reader.

We now consider a variant of Notation 4.

Notation 6. For each \( k \geq 0 \), we let \( \mathcal{D}_k \) denote the category whose objects are finite polyhedra \( E \subseteq \Delta^k \times \mathbb{R}^\infty \) for which the projection \( E \to \Delta^k \) is a fibration, and whose morphisms are cell-like maps \( E \to E' \) which commute with the projection to \( \Delta^k \). We will regard \( \mathcal{D}_\bullet \) as a simplicial category. Let \( \mathcal{D}_\bullet ^o \) denote the “underlying” simplicially enriched category. Ignoring the data of the PL embeddings, we can think of \( \mathcal{D}_\bullet ^o \) as the simplicially enriched category whose objects are finite polyhedra \( K \), where Map\( _{\mathcal{D}_\bullet ^o}(K,K') \) is the simplicial set parametrizing cell-like maps from \( K \) to \( K' \).

Note that we have a canonical isomorphism of simplicial sets \( M \simeq N_0(\mathcal{D}_\bullet ) \). We have the following analogue of Proposition 5:

Proposition 7. The canonical maps

\[ N_\bullet (\mathcal{D}_\bullet ^o) \alpha \hookrightarrow N_\bullet (\mathcal{D}_\bullet ) \leftarrow N_0(\mathcal{D}_\bullet ) = M \]

are weak homotopy equivalences (of bisimplicial sets).

Unlike Proposition 5, Proposition 7 is not a triviality. The first part of the proof breaks down because the inclusions \( \mathcal{D}_k ^o \hookrightarrow \mathcal{D} \) are not equivalences of categories (a PL fibration \( E \to \Delta^k \) need not be a fiber bundle), and the second part of the proof breaks down because cell-like maps need not be invertible. We will give the proof of Proposition 7 in the next lecture. For the moment, let us study its consequences.

For fixed \( d \geq 0 \), we have a commutative diagram (of bisimplicial sets)

\[
\begin{array}{ccc}
N_\bullet (\mathcal{C}_\bullet ^o) & \to & N_\bullet (\mathcal{C}_\bullet ) \\
\downarrow & & \downarrow \theta_d \\
N_\bullet (\mathcal{D}_\bullet ^o) & \to & N_\bullet (\mathcal{D}_\bullet )
\end{array}
\]

Consequently, we can identify \( \theta_d \) with the map of bisimplicial sets \( N_\bullet (\mathcal{C}_\bullet ^o) \to N_\bullet (\mathcal{D}_\bullet ^o) \) induced by the forgetful functor \( \mathcal{C}_\bullet ^o \to \mathcal{D}_\bullet ^o \) (which associates to each framed PL manifold its underlying finite polyhedron).

It follows from general nonsense that the homotopy type of \( N_\bullet (\mathcal{C}_\bullet ^o) \) can be expressed as an iterated homotopy colimit

\[ \text{hocolim}_{K \in \mathcal{D}_\bullet ^o}(\text{hocolim}_{M \in \mathcal{C}_\bullet ^o} \text{Map}_{\mathcal{D}_\bullet ^o}(M,K)). \]
Let us fix an object \( K \in \mathcal{D}_\bullet \) for the moment. It follows from Proposition 5 that we can identify the nerve \( N_\bullet (C_\bullet) \) with the classifying space \( \text{Man}^d \) for framed PL manifolds of dimension \( d \). The homotopy colimit \( \text{Man}_K^d = (\text{hocolim}_{M \in C_\bullet} \text{Map}_{\mathcal{D}^+}(M, K)) \) is equipped with a canonical map

\[
\text{Man}_K^d \to \text{hocolim}_{M \in C_\bullet} \ast \simeq \text{Man}^d,
\]

which is a fibration classified by the functor \( M \mapsto \text{Map}_\mathcal{D}^+(M, K) \). More explicitly, we can identify \( \text{Man}_K^d \) with the simplicial set whose \( k \)-simplices are triples \((E, \rho, q)\), where \((E, \rho)\) is a \( k \)-simplex of \( \text{Man}^d \) and \( q : E \to K \) is a PL map which is cell-like on each fiber of \( E \). The above analysis then gives

\[
\text{Man}_K^d \simeq \text{hocolim}_{\mathcal{D}_\bullet} \text{Man}_K^d.
\]

Note that for each \( d \geq 0 \), the construction \( M \mapsto M \times [0,1] \) determines a stabilization map \( \text{Man}_K^d \to \text{Man}_K^{d+1} \), depending functorially on \( K \). Set \( \text{Man}_K^\infty = \lim_{d \geq 0} \text{Man}_K^d \). It follows from the above analysis that the canonical map

\[
\text{hocolim}_{\mathcal{D}_\bullet} \text{Man}_K^\infty \to \text{Man}^\infty
\]

is a homotopy equivalence. Moreover, the composite map

\[
\text{hocolim}_{\mathcal{D}_\bullet} \text{Man}_K^\infty \to \text{Man}^\infty \xrightarrow{\theta} \lim_{d \geq 0} M
\]

is given by the composition

\[
\text{hocolim}_{\mathcal{D}_\bullet} \text{Man}_K^\infty \to \text{hocolim}_{\mathcal{D}_\bullet} \ast \simeq M \to \lim_{d \geq 0} M,
\]

where the last map is a homotopy equivalence (since the stabilization map \( M \times [0,1] \to M \) is a homotopy equivalence). Consequently, to prove Theorem 3, it will suffice to verify the following:

**Proposition 8.** For every finite polyhedron \( K \), the Kan complex \( \text{Man}_K^\infty \) is contractible.

**Warning 9.** It is natural to try to prove Proposition 8 by showing that the spaces \( \text{Man}_K^d \) become highly connected as \( d \) becomes large. However, this is not necessarily true. For example, take \( K \) to be a single point, so that \( \text{Man}_K^d \) is a classifying space for compact contractible framed PL manifolds. Note that any compact contractible PL \( d \)-manifold \( M \) admits a framing; if \( \text{Man}_K^K \) were connected, then \( M \) would need to be PL homeomorphic to a disk of dimension \( d \). However, this is not necessarily the case: if \( d \geq 6 \), then any closed PL manifold \( B \) with the homology of a \((d-1)\)-sphere bounds a contractible PL manifold \( M \) of dimension \( d \) ([2]). If \( B \) is not simply connected, then \( M \) cannot be homeomorphic to a disk.

Following [3], we introduce an auxiliary condition to rule out the behavior of Warning 9.

**Definition 10.** Let \( K \) be a finite polyhedron. A **d-manifold thickening of \( K \)** is a cell-like PL map \( \pi : M \to K \), where \( M \) is a framed PL manifold of dimension \( d \), having the additional property that for each point \( x \in K \), the intersection \( \partial M \cap \pi^{-1}\{x\} \) is simply connected. We let \( T^d(K) \) denote the simplicial subset of \( \text{Man}_K^d \) whose \( k \)-simplices are triples \((E, \rho, q)\) where \( q : E \to K \) is a \( d \)-manifold thickening of \( K \) on each fiber.

Note that if \( \pi : M \to K \) is any cell-like map and \( \pi' : M \times [0,1] \to K \) is the composition of \( \pi \) with the projection, then \( \partial(M \times [0,1]) \cap \pi'^{-1}\{x\} \) is homotopy equivalent to the suspension of \( \partial(M) \cap \pi^{-1}\{x\} \) for each \( x \in K \). It follows that the stabilization map \( \text{Man}_K^d \to \text{Man}_K^{d+1} \) carries \( T^d(K) \) into \( T^{d+1}(K) \), and that the three-fold iterate of the stabilization map carries all of \( \text{Man}_K^d \) into \( T^{d+3}(K) \). We may therefore identify \( \text{Man}_K^\infty \) with the direct limit \( \lim_{d \geq 0} T^d(K) \). The main step in the proof of Theorem 3 will be to verify the following:

**Proposition 11.** Let \( K \) be a finite polyhedron and let \( n \) be an integer. Then for every sufficiently large integer \( d \) (where the meaning of “sufficiently large” depends on \( K \) and \( n \)), the space of \( d \)-manifold thickenings \( T^d(K) \) is \( n \)-connected.
References

