Let $X$ be a simplicial set. As before, we let $\mathcal{C}_X$ denote the category whose objects are diagrams

$$
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
X & \longrightarrow & X
\end{array}
$$

where $Y$ is obtained from $X$ by adding finitely many simplices. Let $s$ denote the collection of cell-like maps in $\mathcal{C}_X$, let $h$ denote the collection of weak homotopy equivalences in $\mathcal{C}_X$, and let $\mathcal{C}_X^h$ denote the full subcategory of $\mathcal{C}_X$ spanned by those objects where the map $X \rightarrow Y$ is a weak homotopy equivalence. Our goal in this lecture (and the next) is to complete the second part of this course by establishing the following result:

**Proposition 1.** The diagram

$$
\begin{array}{ccc}
K(\mathcal{C}_X^h, s) & \longrightarrow & K(\mathcal{C}_X^h, h) \\
\downarrow & & \downarrow \\
K(\mathcal{C}_X, s) & \longrightarrow & K(\mathcal{C}_X, h)
\end{array}
$$

is a homotopy pullback square.

We will prove Proposition 1 by analyzing the $K$-theory space $K(\mathcal{C}_X, h)$ (which we know to be homotopy equivalent to $\Omega^\infty A^\text{free}(X)$) and eventually showing that it can be identified with the homotopy quotient of $K(\mathcal{C}_X, s)$ by the action of $K(\mathcal{C}_X^h, s)$.

As a first step, it will be convenient to replace $\mathcal{C}_X$ by something slightly closer to $\mathcal{C}_X^h$.

**Definition 2.** For each integer $n$, let $\mathcal{C}_X^{(n)}$ denote the full subcategory of $\mathcal{C}_X$ spanned by those objects for which the map $X \rightarrow Y$ is $n$-connected.

**Lemma 3.** For each integer $n$, the inclusion $\mathcal{C}_X^{(n)} \hookrightarrow \mathcal{C}_X$ induces homotopy equivalences

$$
K(\mathcal{C}_X^{(n)}, s) \rightarrow K(\mathcal{C}_X, s) \quad K(\mathcal{C}_X^{(n)}, h) \rightarrow K(\mathcal{C}_X, h).
$$

**Proof.** We will give the proof of the second assertion; the proof of the first is similar. When $n = -1$, there is nothing to prove. Proceeding by induction on $n$, we are reduced to proving that each of the inclusions $\mathcal{C}_X^{(n+1)} \hookrightarrow \mathcal{C}_X^{(n)}$ induce a homotopy equivalence $K(\mathcal{C}_X^{(n+1)}, h) \rightarrow K(\mathcal{C}_X^{(n)}, h)$. Let $Y$ be an object of $\mathcal{C}_X$, given by a diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow^r & & \downarrow^\text{id} \\
X & \longrightarrow & X
\end{array}
$$
Let \( M(r) = (Y \times \Delta^1) \amalg_{\Gamma_X} X \) denote the mapping cylinder of \( r \) and let \( F(Y) = X \amalg_{\Gamma_Y} M(r) \) denote the two-sided mapping cylinder of \( r \). The construction \( Y \mapsto F(Y) \) induces a functor from \( \mathcal{C}_X \) to itself which carries \( \mathcal{C}_X^{(n)} \) into \( \mathcal{C}_X^{(n+1)} \); in particular, it carries both \( \mathcal{C}_X^{(n)} \) and \( \mathcal{C}_X^{(n+1)} \) to themselves. Note that \( F \) preserves cofibrations, pushouts, weak homotopy equivalences, and cell-like maps. It therefore induces maps on \( K \)-theory. Applying the two-out-of-six property to the diagram of spaces

\[
K(\mathcal{C}_X^{(n+1)}, h) \to K(\mathcal{C}_X^{(n)}, h) \xrightarrow{F} K(\mathcal{C}_X^{(n+1)}, h) \to K(\mathcal{C}_X^{(n)}, h),
\]

we are reduced to showing that \( F \) induces homotopy equivalences from \( K(\mathcal{C}_X^{(n)}, h) \) and \( K(\mathcal{C}_X^{(n+1)}, h) \) to themselves. In fact, we claim that on both \( K \)-theory spaces \( F \) acts by \((-1)\): this follows by applying the additivity theorem to the natural cofiber sequence

\[
Y \to M(r) \to F(Y),
\]

since the functor \( Y \mapsto M(r) \) is related by a cell-like natural transformation to the constant functor \( Y \mapsto X \).

By virtue of Lemma 3, it will suffice to show that the diagram

\[
\begin{array}{ccc}
K(\mathcal{C}_X^{(n)}, h) & \xrightarrow{F} & K(\mathcal{C}_X^{(n)}, h) \\
\downarrow & & \downarrow \\
K(\mathcal{C}_X^{(1)}, h) & \xrightarrow{F} & K(\mathcal{C}_X^{(1)}, h)
\end{array}
\]

is a homotopy pullback square.

Note that \( K(\mathcal{C}_X^{(1)}, h) \) can be obtained as the geometric realization of the simplicial object of \( \text{Set}_{\Delta} \) given by

\[
[n] \mapsto N(hS_n \mathcal{C}_X^{(1)}).
\]

Let us fix \( n \) for the moment, and consider the category \( hS_n \mathcal{C}_X^{(1)} \): the objects of this category can be identified with diagrams

\[
X \leftrightarrow Y_1 \leftrightarrow Y_2 \leftrightarrow \cdots \leftrightarrow Y_n \to X
\]

where all but the last map are 1-connected cofibrations (each adding finitely many simplices) and the composition is the identity, and the morphisms are levelwise weak homotopy equivalences. Let us denote such an object simply by \( \vec{Y} \). We would like to analyze \( hS_n \mathcal{C}_X^{(1)} \) in terms of the subcategory where the morphisms are levelwise cell-like maps. To this end, let us consider a bisimplicial set \( N'(hS_n \mathcal{C}_X^{(1)})_\bullet \bullet \) whose \((p, q)\)-simplices are diagrams

\[
\begin{array}{ccc}
\vec{Y}_{0,0} & \to & \cdots & \to & \vec{Y}_{0,q} \\
\downarrow & & \downarrow & & \downarrow \\
\cdots & \to & \cdots & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
\vec{Y}_{p,0} & \to & \cdots & \to & \vec{Y}_{p,q}
\end{array}
\]

where the horizontal maps are levelwise weak homotopy equivalences and the vertical maps are levelwise cell-like.

**Lemma 4 (Swallowing Lemma).** In the situation above, the canonical map

\[
N(hS_n \mathcal{C}_X^{(1)})_\bullet \to N'(hS_n \mathcal{C}_X^{(1)})_0 \bullet \to N'(hS_n \mathcal{C}_X^{(1)})_\bullet \bullet
\]

is a homotopy equivalence (after geometric realization).
Proof. It will suffice to show that for each \( p \geq 0 \), the natural map \( N'(hS_n C_\mathcal{X})_{p,*} \to N'(hS_n C_\mathcal{X})_{p,*} \) is a weak homotopy equivalence of simplicial sets. Note that the target can be identified with the nerve of the category \( \mathcal{E} \) whose objects are diagrams

\[
\tilde{Y}_0 \to \tilde{Y}_1 \to \cdots \to \tilde{Y}_p
\]

of (levelwise) cell-like maps in \( hS_n C_\mathcal{X}^{(1)} \). The diagonal map \( hS_n C_\mathcal{X}^{(1)} \to \mathcal{E} \) admits a left inverse, given by the construction

\[
\tilde{Y}_0 \to \tilde{Y}_1 \to \cdots \to \tilde{Y}_p \to \tilde{Y}_0.
\]

This left inverse is also a right homotopy inverse by means of the evident natural map

\[
\begin{array}{ccc}
\tilde{Y}_0^2 & \overset{id}{\longrightarrow} & \tilde{Y}_0^2 \\
\downarrow & & \downarrow \\
\tilde{Y}_0 & \longrightarrow & \tilde{Y}_0 \\
\end{array}
\begin{array}{ccc}
\tilde{Y}_0^2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
\tilde{Y}_0 & \longrightarrow & \tilde{Y}_p \\
\end{array}
\begin{array}{ccc}
\tilde{Y}_0^2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
\tilde{Y}_p & \longrightarrow & \tilde{Y}_p \\
\end{array}
\]

It will be convenient to consider a slightly smaller bisimplicial set. We say that a morphism \( \tilde{Y} \to \tilde{Y}' \) in \( hS_n C_\mathcal{X}^{(1)} \) is a cofibration if the induced map \( Y_i' \sqcup_{Y_i} Y_{i+1} \to Y_{i+1}' \) is a monomorphism of simplicial sets for each \( i \). Let \( N''(h C_\mathcal{X})_{*,*} \) denote the bisimplicial set whose objects are diagrams

\[
\begin{array}{ccc}
\tilde{Y}_{0,0} & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
\tilde{Y}_{p,0} & \longrightarrow & \cdots \\
\end{array}
\begin{array}{ccc}
\tilde{Y}_{0,q} & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
\tilde{Y}_{p,q} & \longrightarrow & \cdots \\
\end{array}
\]

where the horizontal maps are cofibrations and levelwise weak homotopy equivalences and the vertical maps are cell-like.

**Lemma 5.** The inclusion of bisimplicial sets

\[
N''(hS_n C_\mathcal{X})_{*,*} \hookrightarrow N'(hS_n C_\mathcal{X})_{*,*}
\]

is a weak homotopy equivalence (after geometric realization).

**Proof.** It will suffice to show that for each integer \( p \geq 0 \), the inclusion

\[
N''(hS_n C_\mathcal{X})_{p,*} \hookrightarrow N'(hS_n C_\mathcal{X})_{p,*}
\]

is a weak homotopy equivalence. In other words, if we let \( \mathcal{E}_0 \subseteq \mathcal{E} \) be the subcategory of \( \mathcal{E} \) whose morphisms are given by diagrams

\[
\begin{array}{ccc}
\tilde{Y}_0^2 & \overset{id}{\longrightarrow} & \tilde{Y}_1^2 \\
\downarrow & & \downarrow \\
\tilde{Y}_0 & \longrightarrow & \tilde{Y}_1 \\
\end{array}
\begin{array}{ccc}
\tilde{Y}_0^2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
\tilde{Y}_0 & \longrightarrow & \tilde{Y}_p \\
\end{array}
\begin{array}{ccc}
\tilde{Y}_0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow \\
\tilde{Y}_0 & \longrightarrow & \tilde{Y}_p \\
\end{array}
\]

is a weak homotopy equivalence.
where the vertical maps are cofibrations (as well as being weak homotopy equivalences), then we wish to show that the inclusion $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ is a weak homotopy equivalence. Let us assume for simplicity that $p = n = 0$ (the proof in the general case is differs only by notation): then $\mathcal{E}$ is the subcategory of $\mathcal{C}^{(1)}_{X}$ whose morphisms are weak homotopy equivalences, and $\mathcal{E}_0$ is the subcategory of $\mathcal{C}^{(1)}_{X}$ whose morphisms are trivial cofibrations. We will prove that the inclusion $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ is a weak homotopy equivalence by showing that it is right cofinal. To this end, fix an object $Y \in \mathcal{E}$; we wish to show that the category $\mathcal{D} = \mathcal{E}_0 \times \mathcal{E}/Y$ is weakly contractible. Unwinding the definitions, we can identify the objects of $\mathcal{D}$ are weak homotopy equivalences, and where the vertical maps are cofibrations (as well as being weak homotopy equivalences), then we wish to of trivial cofibrations

$$Y' \hookrightarrow (M(f) \amalg_\Delta^1 X) \hookrightarrow Y$$

where $M(f) = (Y' \times \Delta^1) \amalg_{Y' \times \{1\}} Y$ is the mapping cylinder of $f$.

Let us now reorganize a bit. For each $q \geq 0$, let $F_q(\mathcal{C}^{(1)}_{X})$ denote the category whose objects are sequences of trivial cofibrations

$$Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_q$$

in $\mathcal{C}^{(1)}_{X}$. Then we can regard $F_q(\mathcal{C}^{(1)}_{X})$ as a category with cofibrations (defined as above, with the roles of $n$ and $q$ switched) and weak equivalences (given by the collection $s$ of levelwise cell-like maps). This category with cofibrations and weak equivalences depends functorially on $[q]$, so we can regard $\mathcal{F}_q \mathcal{C}^{(1)}_{X}$ as a simplicial category with cofibrations and weak equivalences. Unwinding the definitions, we have

$$K(F_q(\mathcal{C}^{(1)}_{X})) \simeq |N(hS\mathcal{F}_q(\mathcal{C}^{(1)}_{X}))|.$$ 

Passing to the geometric realization as $[q]$ varies and invoking Lemmas 4 and 5, we obtain a homotopy equivalence

$$K(\mathcal{C}^{(1)}_{X}, h) \simeq |K(\mathcal{F}_q \mathcal{C}^{(1)}_{X}, s)|.$$

Given a cofibration $Y \hookrightarrow Y'$ in $\mathcal{C}^{(1)}_{X}$, let $Y'/Y$ denote the pushout $Y' \amalg_{Y} X$. It is clear that if $Y \hookrightarrow Y'$ is a weak homotopy equivalence, then the quotient $Y'/Y$ is weakly homotopy equivalent to $X$. If $Y'$ and $Y$ both belong to $\mathcal{C}^{(1)}_{X}$, then the converse holds: this follows from the observation that for any local system of abelian groups $A$ on $X$, we have an isomorphism

$$H_*(Y', Y; A|_{Y'}) \simeq H_*(Y'/Y, X; A|_{Y'/Y}).$$

It follows that $F_q(\mathcal{C}^{(1)}_{X})$ admits an alternative description: it can be identified with the category whose objects are sequences of cofibrations

$$Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_q$$

in $\mathcal{C}^{(1)}_{X}$ where each quotient $Y_i/Y_{i-1}$ belongs to $\mathcal{C}^h_{X}$.

There is a natural map

$$\theta_q : (\mathcal{C}^{(1)}_{X} \times (\mathcal{C}^{h}_{X})^q) \to \mathcal{F}_q(\mathcal{C}^{(1)}_{X}),$$

given on objects by

$$(Y; Z_1, \ldots, Z_q) \mapsto (Y \hookrightarrow Y \amalg_{Y} Z_1 \hookrightarrow \cdots \hookrightarrow Y \amalg_{Y} Z_1 \amalg_{Y} Z_1 \cdots \amalg_{Y} Z_q)$$

This induces a map on $K$-theory spaces

$$K(\mathcal{C}^{(1)}_{X}, s) \times K(\mathcal{C}^{h}_{X}, s)^q \to K(F_q(\mathcal{C}^{(1)}_{X}), s).$$

Passing to the geometric realization as $q$ varies, we obtain a map

$$K(\mathcal{C}^{(1)}_{X}, s)/K(\mathcal{C}^{h}_{X}, s)^q \to |K(\mathcal{F}_q \mathcal{C}^{(1)}_{X}, s)| \simeq K(\mathcal{C}^{(1)}_{X}, h).$$
To prove Proposition 1, it will suffice to show that this map is a homotopy equivalence. In fact, we will prove something stronger: each of the maps $\theta_q$ induces a homotopy equivalence at the level of $K$-theory. Note that $\theta_q$ has a left homotopy inverse $\rho$, given by the construction

$$(Y_0 \hookrightarrow \cdots \hookrightarrow Y_q) \mapsto (Y_0, (Y_1/Y_0, \cdots, Y_q/Y_{q-1})).$$

The composition $\theta_q \circ \rho$ is not homotopic to the identity at the level of categories, but induces the identity map on $K$-theory spaces (up to homotopy) by virtue of the additivity theorem.

References