The Whitehead Space II (Lecture 31)

November 19, 2014

Let $X$ be a simplicial set. As in the previous lecture, we let $D_X$ denote the subcategory of $(\text{Set}_\Delta)_X$ spanned by those objects $i : X \rightarrow Y$ which are trivial cofibrations of simplicial sets obtained by adding finitely many simplices to $X$, and whose morphisms are cell-like maps. We let $W(X)$ denote the nerve of $D_X$. Our goal in this lecture is to show that if $X$ is finite, then $W(X)$ can be identified with the homotopy fiber product $M \times_{M^h} \{X\}$. Our first step is to establish the following result (already used without proof in the previous lecture):

**Proposition 1.** The functor $X \mapsto W(X)$ preserves weak homotopy equivalences.

We will deduce Proposition 1 from two special cases:

**Lemma 2.** Let $X$ be a finite simplicial set. Then the “last vertex” map $Sd(X) \rightarrow X$ induces a weak homotopy equivalence $W(Sd(X)) \rightarrow W(X)$.

**Lemma 3.** Let $X$ be a finite simplicial set. Then the projection map $X \times \Delta^1 \rightarrow X$ induces a weak homotopy equivalence $W(X \times \Delta^1) \rightarrow W(X)$.

**Proof of Proposition 1.** We first show that if $f : X \rightarrow Y$ is a weak homotopy equivalence of finite simplicial sets, then the induced map $W(X) \rightarrow W(Y)$ is a weak homotopy equivalence. Since $X$ is not a Kan complex, the map $f$ need not be a homotopy equivalence. However, there exists a homotopy inverse to $f$ after fibrant replacement: that is, a map $g : Y \rightarrow Ex^\infty X$ such that the unit map $X \rightarrow Ex^\infty X$ is homotopic to $g \circ f$, and $Ex^\infty(f) \circ g$ is homotopic to the unit map $Y \rightarrow Ex^\infty Y$. Since $X$ and $Y$ are finite, we can replace $Ex^\infty$ by $Ex^n$ for $n \gg 0$. In this case, we can identify $g$ with a map $G : Sd^n(Y) \rightarrow X$, and we have homotopies $h : Sd^n(X \times \Delta^1) \rightarrow X$ and $h' : Sd^n(Y \times \Delta^1) \rightarrow Y$.

To show that these maps yield homotopies after applying $W$, it suffices to show that the maps

$W(Sd^n(X \times \Delta^1)) \rightarrow W(X)$

$W(Sd^n(X)) \rightarrow W(X)$

are weak homotopy equivalences, and similarly for $Y$; these assertions are immediate consequences of Lemmas 2 and 3.

It follows from the above argument that when restricted to finite simplicial sets, the functor $W : \text{Set}_\Delta \rightarrow S$ preserves weak homotopy equivalences, and therefore induces a functor of $\infty$-categories $u : S^{\text{fin}} \rightarrow S$. The functor $u$ admits an essentially extension $U : S \rightarrow S$ which commutes with filtered colimits. Since $W$ commutes with filtered colimits, it follows that it is given by the composition

$\text{Set}_\Delta \rightarrow S [\rightarrow S]$. 

$\square$
Proof of Lemma 3. Pushout along the projection map \( X \times \Delta^1 \to X \) induces a functor \( f : \mathcal{D}_{X \times \Delta^1} \to \mathcal{D}_X \). Consider the functor \( g : \mathcal{D}_X \to \mathcal{D}_{X \times \Delta^1} \) given by \( Y \mapsto Y \times \Delta^1 \). We claim that, after passing to nerves, these maps are mutually inverse homotopy equivalences relating \( W(X \times \Delta^1) \) and \( W(X) \). Note that \( f \circ g : \mathcal{D}_X \to \mathcal{D}_X \) is the functor given by
\[
Y \mapsto X \amalg_{X \times \Delta^1} (Y \times \Delta^1).
\]
At the level of nerves, this is homotopic to the identity map, since the projection \( Y \times \Delta^1 \to Y \) induces a cell-like map
\[
X \amalg_{X \times \Delta^1} (Y \times \Delta^1) \to Y.
\]
The functor \( g \circ f : \mathcal{D}_{X \times \Delta^1} \to \mathcal{D}_{X \times \Delta^1} \) is given by
\[
Y \mapsto (Y \amalg_{X \times \Delta^1} X) \times \Delta^1.
\]
In this case, we have a two-step homotopy to the identity, given by the diagram
\[
(Y \amalg_{X \times \Delta^1} X) \times \Delta^1 \leftarrow Y \times \Delta^1 \to Y.
\]
\[\square\]

Proof of Lemma 2. We wish to show that the functor
\[
f : \mathcal{D}_{\text{Sd}(X)} \to \mathcal{D}_X
\]
\[
Y \mapsto Y \amalg_{\text{Sd}(X)} X
\]
induces a weak homotopy equivalence on nerves. We will show that the construction
\[
g : \mathcal{D}_X \to \mathcal{D}_{\text{Sd}(X)}
\]
\[
Y \mapsto \text{Sd}(Y)
\]
provides a homotopy inverse. Note that the composite map \( f \circ g : \mathcal{D}_X \to \mathcal{D}_X \) is related to the identity functor by a cell-like natural transformation
\[
\text{Sd}(Y) \amalg_{\text{Sd}(X)} X \to Y.
\]
The other direction is a bit trickier: the composite functor \( g \circ f : \mathcal{D}_{\text{Sd}(X)} \to \mathcal{D}_{\text{Sd}(X)} \) carries an object \( Y \in \mathcal{D}_{\text{Sd}(X)} \) to the object \( \text{Sd}(Y) \amalg_{\text{Sd}(X)} \text{Sd}(X) \) under the functor \( \mathcal{D}_{\text{Sd}^2(X)} \to \mathcal{D}_{\text{Sd}(X)} \), which is obtained from the map \( \text{Sd}(e) : \text{Sd}^2(X) \to \text{Sd}(X) \), where \( e : \text{Sd}(X) \to X \) is the “last vertex map”. Note that \( e \) does not coincide with the “last vertex” map \( \text{Sd}^2(X) \to \text{Sd}(X) \), but it is simplicially homotopic to it, and therefore (by virtue of Lemma 3) induces a homotopic map from \( W(\text{Sd}^2(X)) \) to \( W(\text{Sd}(X)) \). We are therefore reduced to proving that the functor \( Y \mapsto \text{Sd}(Y) \amalg_{\text{Sd}^2(X)} \text{Sd}(X) \) is homotopic to the identity, where \( \text{Sd}^2(X) \) maps to \( \text{Sd}(X) \) via the “last vertex” map. This follows from the first part of the proof (applied to \( \text{Sd}(X) \) rather than \( X \)).
\[\square\]

It will be useful for us to consider a slight variant of the category \( \mathcal{D}_X \). From this point forward, let us assume that the simplicial set \( X \) is finite. Let \( \mathcal{D}_X^+ \) denote the subcategory of \( (\text{Set}_\Delta)_X \), whose objects are weak homotopy equivalences \( X \to Y \) of finite simplicial sets, and whose morphisms are cell-like maps. Then \( \mathcal{D}_X^+ \) contains \( \mathcal{D}_X \) as a full subcategory: the only difference is that we no longer require the structure map \( X \to Y \) to be a colibration.

Proposition 4. For every finite simplicial set \( X \), the inclusion \( \mathcal{D}_X \hookrightarrow \mathcal{D}_X^+ \) induces a weak homotopy equivalence of nerves.
Proof. For each morphism \( f : X \to Y \), let \( M(f) = (X \times \Delta^1) \amalg_{X \times \{1\}} Y \) denote the mapping cylinder of \( f \). Then the construction

\[(f : X \to Y) \mapsto (X \times \{0\} \amalg M(f))\]

determines a functor from \( D_X^+ \) into \( D_X \). Using the natural cell-like map \( M(f) \to Y \), we see that this functor determines a deformation retraction of \( N(D_X^+) \) into \( N(D_X) \).

Note that the enlargement \( D_X \mapsto D_X^+ \) comes at a price: if \( X \to X' \) is a map of finite simplicial sets, the construction

\[Y \mapsto Y \amalg X \amalg X'\]

generally does not preserve cell-like maps (or weak homotopy equivalences), and therefore does not induce a functor from \( D_X^+ \) to \( D_{X'}^+ \). However, we get a different sort of functoriality as compensation: if \( f : X \to X' \) is a weak homotopy equivalence, then composition with \( f \) induces a map \( D_X^+ \to D_{X'}^+ \). We will need the following variant of Proposition 1:

**Proposition 5.** Let \( f : X \to X' \) be a weak homotopy equivalence of finite simplicial sets. Then composition with \( f \) induces a weak homotopy equivalence \( D_X^+ \to D_{X'}^+ \).

Proof. Arguing as in the proof of Proposition 1, it suffices to treat the case of the maps

\[Sd(X) \to X \quad X \times \Delta^1 \to X.\]

Using Propositions 1 and 4, we are reduced to proving the composite functor

\[D_X \to D_{X'} \to D_{X'}^+ \to D_X^+\]

\[Y \mapsto Y \amalg X \amalg X'\]

is a weak homotopy equivalence. Since \( f \) is cell-like, this functor is related to the inclusion \( D_X \mapsto D_X^+ \) by a natural transformation \( Y \to Y \amalg X \amalg X' \); the desired result now follows from Proposition 4.

Fix a finite simplicial set \( X \). For each \( n \geq 0 \), the construction \([m] \mapsto Sd(X \times \Delta^m)\) determines a cosimplicial object of \( \mathcal{C} \). We therefore obtain a simplicial category

\[D_{Sd^n(X \times \Delta^*)}^+ .\]

After taking nerves, we obtain a simplicial space which is equivalent to the constant simplicial space with the value

\[N(D_{Sd^n(X \times \Delta^*)}^+) \simeq N(D_X^+) \simeq N(D_X) \simeq W(X).\]

We may therefore identify \( W(X) \) with the geometric realization

\[| \lim_{n \to \infty} N D_{Sd^n(X \times \Delta^*)}^+ |.\]

Let \( \mathcal{E} \) denote the category whose objects are finite simplicial sets \( Y \) and whose morphisms are cell-like maps. Then each of the categories \( D_{Sd^n(X \times \Delta^m)}^+ \) is cofibered in sets over \( \mathcal{E} \), and can therefore be identified with the Grothendieck construction on the functor

\[f_{n,m} : \mathcal{E} \to Set\]

which assigns to each object \( Y \in \mathcal{E} \) the set of all weak homotopy equivalences

\[Sd^n(X \times \Delta^m) \to Y.\]
It follows that the nerve of $D_{sd^n(X \times \Delta_\infty)}^+$ can be identified with the homotopy colimit of the diagram $f_{n,m}$. It follows that
\[
\lim_n N D_{sd^n(X \times \Delta_\infty)}^+
\]
can be identified with the homotopy colimit of the functor
\[
f_m : E \to \text{Set}
\]
\[
f_m(Y) = \text{Hom}'(X \times \Delta^m, \text{Ex}^\infty Y).
\]
where $\text{Hom}'(X \times \Delta^m, \text{Ex}^\infty Y)$ is the subset of $\text{Hom}(X \times \Delta^m, \text{Ex}^\infty Y)$ consisting of weak homotopy equivalences. Passing to the geometric realization, we can identify $W(X)$ with the homotopy colimit of the diagram
\[
\mathcal{E} \to \text{Set}_\Delta
\]
\[
Y \mapsto H(X, \text{Ex}^\infty Y)
\]
where $H(X, \text{Ex}^\infty Y)$ is the simplicial set parametrizing homotopy equivalences from $X$ to $\text{Ex}^\infty Y$. We saw in Lecture 12 that we can identify $\mathcal{M}$ with the nerve of $\mathcal{E}$; this identification induces an equivalence
\[
W(X) \simeq \lim_{Y \in \mathcal{E}} H(X, \text{Ex}^\infty Y) \simeq N(\mathcal{E}) \times_{\mathcal{M}^h} \{X\} \simeq M \times_{\mathcal{M}^h} \{X\}.
\]

We conclude by discussing the extent to which the homotopy equivalence $W(X) \simeq M \times_{\mathcal{M}^h} \{X\}$ can be made functorial in $X$. By virtue of the above discussion, this amounts to the question of how functorially we can identify the spaces $N(D_X)$ with $N(D_X^+)$.

**Proposition 6.** The functor $u$ and $v$ are homotopic to one another (after identifying $S^\infty$ with its opposite).

**Warning 7.** We can extend $u$ and $v$ to functors defined on the larger category whose objects are finite simplicial sets and whose morphisms are weak homotopy equivalences. However, these enlargements are not equivalent to one another (note that if they were, then the fibration $\mathcal{M} \to \mathcal{M}^h$ would be classified by the functor $X \mapsto W(X)$, and would therefore admit a section).

To prove Proposition 6, we begin by applying the Grothendieck construction to the assignments
\[
X \mapsto D_X \quad X \mapsto D_X^+
\]
to produce coCartesian fibrations
\[
D \to \mathcal{E} \quad D^+ \to \mathcal{E}^{op}:
\]
the objects of $D$ are trivial cofibrations $i : X \to Y$ of finite simplicial sets, and the objects of $D^+$ are weak homotopy equivalences $i : X \to Y$ of finite simplicial sets. Morphisms in $D$ are given by commutative diagrams
\[
\begin{array}{c}
X \longrightarrow Y \\
\downarrow \quad \downarrow \\
X' \longrightarrow Y'
\end{array}
\]
where the vertical maps are cell-like and the horizontal maps are trivial cofibrations, and morphisms in \( \mathcal{D}^+ \) are given by commutative diagrams

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\uparrow & & \uparrow \\
X' & \longrightarrow & Y'
\end{array}
\]

where the vertical maps are cell-like and the horizontal maps are weak homotopy equivalences.

**Remark 8.** There is a bit of work hidden in this description of \( \mathcal{D} \). *A priori*, the morphisms in the relevant Grothendieck construction are given by commutative diagrams

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow^f & & \downarrow^g \\
X' & \longrightarrow & Y'
\end{array}
\]

where the horizontal maps are trivial cofibrations, \( f \) is cell-like, and the induced map \( g' : Y \amalg_X X' \to Y' \) is cell-like. But the assumption that \( f \) is cell-like guarantees that the map \( Y \to Y \amalg_X X' \) is cell-like, from which it follows that \( g \) is cell-like if and only if \( g' \) is cell-like.

These coCartesian fibrations induce maps of spaces

\[
U : N(\mathcal{D}) \to N(\mathcal{E}) \quad V : N(\mathcal{D}^+) \to N(\mathcal{E}^{\text{op}}).
\]

Using Propositions 1 and 5 and Quillen’s Theorem B, we see that the homotopy fibers of these maps (over an object \( X \in \mathcal{E} \)) can be identified with \( N(\mathcal{D}_X) \) and \( N(\mathcal{D}_X^+) \), respectively. Consequently, Proposition 6 can be reformulated as follows: the natural homotopy equivalence \( N(\mathcal{E}) \simeq N(\mathcal{E}^{\text{op}}) \) can be lifted to an equivalence between \( U \) and \( V \) (regarded as objects in the \( \infty \)-category \( \text{Fun}(\Delta^1, \mathcal{S}) \) of morphisms in the \( \infty \)-category of spaces).

To prove this, let \( \text{TwArr}(\mathcal{E}) \) denote the “twisted arrow category” of \( \mathcal{E} \): that is, the category whose objects are weak homotopy equivalences \( f : X_0 \to X_1 \) of finite simplicial sets, where a morphism from \( f : X_0 \to X_1 \) to \( f' : X'_0 \to X'_1 \) is a commutative diagram

\[
\begin{array}{ccc}
X_0 & \longrightarrow & X_1 \\
\downarrow^{f} & & \downarrow^{g} \\
X'_0 & \longrightarrow & X'_1
\end{array}
\]

\( (f : X_0 \to X_1) \to (X_0, X_1) \) determines a coCartesian fibration

\[
\text{TwArr}(\mathcal{E}) \to \mathcal{E}^{\text{op}} \times \mathcal{E}.
\]

In particular, we have coCartesian fibrations

\[
\mathcal{E}^{\text{op}} \xleftarrow{\epsilon_0} \text{TwArr}(\mathcal{E}) \xrightarrow{\epsilon_1} \mathcal{E}
\]

The fibers of these coCartesian fibrations are weakly contractible (since they have initial objects), so Quillen’s Theorem B implies that \( \epsilon_0 \) and \( \epsilon_1 \) are weak homotopy equivalences; the diagram of spaces

\[
N(\mathcal{E}^{\text{op}}) \leftarrow N(\text{TwArr}(\mathcal{E})) \to N(\mathcal{E})
\]

supplies a concrete combinatorial description of the natural equivalence between \( N(\mathcal{E}^{\text{op}}) \) and \( N(\mathcal{E}) \) (in the \( \infty \)-category of spaces). We may therefore reformulate Proposition 6 as follows: the spaces \( N(\mathcal{D}^+ \times_{\mathcal{E}^{\text{op}}} \text{TwArr}(\mathcal{E})) \)
and $N(\mathcal{D} \times \mathcal{E}\text{TwArr}(\mathcal{E}))$ are equivalent (in the $\infty$-category of spaces over $\text{TwArr}(\mathcal{E})$). Note that we can identify the objects of $\mathcal{D} \times \mathcal{E}\text{TwArr}(\mathcal{E})$ with diagrams of finite simplicial sets $X_0 \xrightarrow{f} X_1 \xrightarrow{g} Y$ where $f$ is a weak homotopy equivalence and $g$ is a trivial cofibration, and we can identify the objects of $\mathcal{D}^+ \times \mathcal{E}\text{TwArr}(\mathcal{E})$ with diagrams

$$Y \xleftarrow{h} X_0 \xrightarrow{f} X_1$$

where $f$ and $h$ are weak homotopy equivalences. The construction $(f, g) \mapsto (f, g \circ f)$ determines a functor

$$\mathcal{D} \times \mathcal{E}\text{TwArr}(\mathcal{E}) \to \mathcal{D}^+ \times \mathcal{E}\text{TwArr}(\mathcal{E})$$

compatible with the projection to $\text{TwArr}(\mathcal{E})$.

It will therefore suffice to show that this functor is a weak homotopy equivalence. To prove this, it suffices to show that it induces an equivalence on homotopy fibers taken over any point $(f : X_0 \to X_1) \in \text{TwArr}(\mathcal{E})$. Unwinding the definitions, we wish to show that composition with $f$ induces a weak homotopy equivalence

$$\mathcal{D}_{X_1} \to \mathcal{D}^+_{X_0},$$

this follows from Propositions 5 and 4.