Another Model of the Assembly Map II (Lecture 29)

November 12, 2014

Let $X$ be a simplicial set. In the previous lecture, we introduced the category $\mathcal{C}_X$ whose objects are simplicial sets $Y$ over and under $X$, which are obtained from $X$ by adding finitely many simplices. We can regard $\mathcal{C}_X$ as a category with cofibrations and weak equivalences (where the latter is given by the collection $s$ of cell-like maps), and we proved that the construction

$$F(X) = \Omega^{-\infty}K(\mathcal{C}_X, s)$$

has the property that $\hat{F}(X) = |F(X\Delta^*)|$ is a homology theory (that is, it induces a colimit-preserving functor from spaces to spectra). There is a natural map $\hat{F}(X) \to A(X)$; when $X$ is finite and nonsingular, this comes from a composite map

$$F(X) \to \Omega^{-\infty}K(\text{Shv}_{PL}(X)^{op}) \to \Omega^{-\infty}K(\Delta) \to A(X)$$

where the first map is obtained from a functor

$$\lambda : \mathcal{C}_X \to \text{Shv}_{PL}(|X|)^{op}$$

which assigns to each retraction diagram

\[
\begin{array}{cc}
Y & \overset{r}{\to} X \\
\searrow & \downarrow \text{id} \\
X & \to X
\end{array}
\]

the constructible sheaf given by the cofiber of the unit map $S_X \to r_*S_Y$. To verify that this construction yields a well-defined map on K-theory, we observe that if $Y, Y' \in \mathcal{C}_X$ are related by a cell-like map $Y \to Y'$, then the constant sheaf $S_{Y'}$ can be identified with the direct image of the constant sheaf $S_Y$, so that the induced map $\lambda(Y') \to \lambda(Y)$ is an equivalence in $\text{Shv}_{PL}(|X|)$.

Consider the functor $\mu : \text{Shv}_{PL}(|X|)^{op} \to (\text{Sp}^X)^{c}$ (here $(\text{Sp}^X)^{c}$ denotes the $\infty$-category of compact objects of $\text{Sp}^X$) characterized by the formula

$$\text{Map}_{\text{Sp}^X}(\mu(\mathcal{F}), \mathcal{G}) = \Gamma(|X|, \mathcal{F} \wedge \mathcal{G}).$$

Unwinding the definitions, we see that $\mu \circ \lambda : \mathcal{C}_X \to (\text{Sp}^X)^{c}$ is given by the formula $Y \mapsto r_!S_Y$, where $r_! : \text{Sp}^Y \to \text{Sp}^X$ is the homological pushforward on local systems. Here we have a bit more flexibility: in order to ensure that a map $Y \to Y'$ in $\mathcal{C}_X$ induces an equivalence $(\mu \circ \lambda)(Y) \to (\mu \circ \lambda)(Y')$ in $\text{Sp}^X$, it is sufficient to assume that $Y \to Y'$ is a weak homotopy equivalence.

**Exercise 1.** Let $h$ be the collection of all weak homotopy equivalences in $\mathcal{C}_X$. Show that $(\mathcal{C}_X, h)$ satisfies the axioms for a category with cofibrations and weak equivalences (where the cofibrations, as before, are given by the monomorphisms).
The above analysis supplies a diagram of infinite loop spaces

\[
\begin{array}{ccc}
K(\mathcal{E}_X, s) & \longrightarrow & K(\mathcal{E}_X, h) \\
\downarrow & & \downarrow \phi \\
K^\Delta(X) & \longrightarrow & \Omega^\infty A(X)
\end{array}
\]

which commutes up to canonical homotopy and depends functorially on \(X\). Our goal in this lecture is to show that the right vertical map is close to being a homotopy equivalence. To this end, recall that \((\text{Sp}^X)^c\) can be identified with the Spanier-Whitehead \(\infty\)-category of the \(\infty\)-category \((\mathcal{S}^X)^c \simeq \mathcal{S}_X/\!/X\), where \(\mathcal{S}_X/\!/X\) is the full subcategory of \(\mathcal{S}_X/\!/X\) spanned by the compact objects. Let \(\mathcal{S}^\text{fin}_X/\!/X \subseteq \mathcal{S}_X/\!/X\) denote the full subcategory spanned by those objects \(Y\) which can be obtained from \(X\) by attaching finitely many cells. Then \(\theta\) is the map on \(K\)-theory induces by the composition

\[\mathcal{E}_X \to \mathcal{S}^\text{fin}_X/\!/X \subseteq \mathcal{S}_X/\!/X \to (\text{Sp}^X)^c.\]

We have seen that the map \(K(\mathcal{S}^\text{fin}_X/\!/X) \to K((\text{Sp}^X)^c) \simeq \Omega^\infty A(X)\) is a homotopy equivalence, and that the map \(K(\mathcal{S}^\text{fin}_X/\!/X) \to K(\mathcal{S}_X/\!/X)\) exhibits the domain as a union of connected components of the target.

**Notation 2.** Let \(X\) be a space. We let \(A^\text{free}(X)\) denote the spectrum given by \(\Omega^{-\infty}K(\mathcal{S}^\text{fin}_X/\!/X)\). Then \(A^\text{free}(X)\) is a connective spectrum whose homotopy groups are given by

\[\pi_i A^\text{free}(X) = \begin{cases} 
\pi_i A(X) & \text{if } i > 0 \\
H_0(X; \mathbb{Z}) & \text{if } i = 0.
\end{cases}\]

(note: this is the definition of \(A(X)\) that appears in Waldhausen’s paper).

Our next goal is to prove:

**Proposition 3.** Let \(X\) be a simplicial set. Then the natural map \(\mathcal{E}_X \to \mathcal{S}^\text{fin}_X/\!/X\) induces homotopy equivalences

\[K(\mathcal{E}_X, h) \to K(\mathcal{S}^\text{fin}_X/\!/X)\]

\[\Omega^{-\infty}K(\mathcal{E}_X, h) \to A^\text{free}(X).\]

Note that the domain and codomain of the map appearing in the statement of Proposition 3 can be identified with the geometric realization of simplicial spaces obtained from Waldhausen’s construction. It will therefore suffice to show that that the map \(\mathcal{E}_X \to \mathcal{S}^\text{fin}_X/\!/X\) induces an equivalence in each simplicial degree. In other words, Proposition 3 is a consequence of the following more precise assertion:

**Proposition 4.** Let \(X\) be a simplicial set and let \(n \geq 0\) be an integer. Then the natural map

\[hS_n \mathcal{E}_X \to S_n \mathcal{S}^\text{fin}_X/\!/X\]

is a weak homotopy equivalence (here we regard the left hand side as the nerve of a category and the right hand side as a Kan complex).

The proof of Proposition 4 will proceed by induction on \(n\). The case \(n = 0\) is trivial (since both sides are contractible), so let us assume \(n > 0\). Let us identify the objects of \(hS_n \mathcal{E}_X\) with chains of cofibrations

\[Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots \hookrightarrow Y_n\]
of simplicial sets over and under $X$. There is a canonical map $\tau : hS_n \mathbb{C}_X \to hS_{n-1} \mathbb{C}_X$ obtained by “forgetting” $Y_n$ (one of the face maps appearing in the simplicial category $hS_\bullet \mathbb{C}_X$). Similarly, we have a forgetful map $S_n \mathbb{S}^\text{fin}_{X//X} \to S_{n-1} \mathbb{S}^\text{fin}_{X//X}$ which is a Kan fibration. These maps fit into a commutative diagram

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
hS_n \mathbb{C}_X & \rightarrow & S_n \mathbb{S}^\text{fin}_{X//X} \\
\downarrow \tau & & \downarrow \\
hS_{n-1} \mathbb{C}_X & \rightarrow & S_{n-1} \mathbb{S}^\text{fin}_{X//X}
\end{array}
\end{array}
\end{array}
$$

The inductive hypothesis implies that the lower horizontal map is a weak homotopy equivalence. Consequently, to prove Proposition 4, it will suffice to show that the diagram induces a weak homotopy equivalence after taking homotopy fibers in the vertical direction. Fix an object $\vec{Y} = (Y_1 \to \cdots \to Y_{n-1})$ in $hS_{n-1} \mathbb{C}_X$. We will show that the category $(hS_n \mathbb{C}_X) \times (hS_{n-1} \mathbb{C}_X) \{\vec{Y}\}$ is weakly homotopy equivalent to the Kan complex $S_n \mathbb{S}^\text{fin}_{X//X} \times S_{n-1} \mathbb{S}^\text{fin}_{X//X} \{\vec{Y}\}$.

It will then follow that every map $\vec{Y} \to \vec{Y}'$ in $hS_{n-1} \mathbb{C}_X$ induces a weak homotopy equivalence $(hS_n \mathbb{C}_X) \times (hS_{n-1} \mathbb{C}_X) \{\vec{Y}\} \to (hS_n \mathbb{C}_X) \times (hS_{n-1} \mathbb{C}_X) \{\vec{Y}'\}$.

Applying Quillen’s Theorem B (and the observation that $\tau$ is a coCartesian fibration), it follows that $(hS_n \mathbb{C}_X) \times (hS_{n-1} \mathbb{C}_X) \{\vec{Y}\}$ can be identified with the homotopy fiber of $\tau$ over $\vec{Y}$, thereby completing the proof of the inductive step. We are therefore reduced to proving the following lemma (applied in the case $Z = Y_{n-1}$):

**Lemma 5.** Let $f : Z \to X$ be a map of simplicial sets. Let $\mathbb{C}_f$ denote the category whose objects are diagrams of simplicial sets

$$
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
Z & \rightarrow & W \\
j & & q
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
Y & \rightarrow & X \\
f & & \phantom{q}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

where $j$ is a cofibration and $Y$ is obtained from $Z$ by adding only finitely many simplices, and whose morphisms are weak homotopy equivalences. Let $\mathbb{S}^\text{fin}_{Z//X}$ denote the $\infty$-category given by the full subcategory of $\mathbb{S}_{Z//X}$ spanned by those objects $Y$ which can be obtained from $Z$ by attaching finitely many cells. Then the canonical map

$$
\nu : \mathbb{C}_f \to (\mathbb{S}^\text{fin}_{Z//X})^\simeq
$$

is a weak homotopy equivalence of simplicial sets.

**Proof.** Let us compute the homotopy fiber of $\nu$ over a point $\eta \in (\mathbb{S}^\text{fin}_{Z//X})^\simeq$. Let us represent $\eta$ by a diagram of simplicial sets

$$
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
W & \rightarrow & X \\
j & & q
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
Z & \rightarrow & X \\
f & & \phantom{q}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

where $j$ is a cofibration and $q$ is a Kan fibration. Then the homotopy fiber $\nu^{-1} \{\eta\}$ can be identified with the homotopy colimit

$$
\lim_{Y \in \mathbb{C}_f} \mathbb{Hom}(Y, W),
$$

3
where $\text{Hom}(Y, W)$ denotes the Kan complex parametrizing maps from $Y$ to $W$ in $(\text{Set}_\Delta)_{Z/X}$ which are weak homotopy equivalences. It follows that $\nu^{-1}\{\eta\}$ can be identified with the geometric realization of a simplicial space which is given in degree $m$ by the homotopy colimit

$$\lim_{Y \in C} \text{Hom}(Y, W)_m.$$ 

It will therefore suffice to show that this homotopy colimit is contractible for each $m$. Replacing $W$ by $W \Delta^m \times_{\Delta^m} X$, we can reduce to the case where $m = 0$. In this case, the homotopy colimit can be identified with the nerve of the category $\mathcal{D}$ whose objects are commutative diagrams

$$\begin{array}{ccc}
Y & \xrightarrow{f} & W \\
\downarrow & & \downarrow \\
Z & \rightarrow & W
\end{array}$$

where $Y$ is obtained from $Z$ by adjoining finitely many simplices and the map $f$ is a homotopy equivalence. It will therefore suffice to show that $\mathcal{D}$ is weakly contractible. In fact, we claim that $\mathcal{D}$ is filtered. Note that $\mathcal{D}$ is a full subcategory of a filtered category $\mathcal{D}^+$, where we drop the requirement that the map $f$ be a weak homotopy equivalence. To prove that $\mathcal{D}$ is also filtered, it will suffice to verify that for every object $Y \in \mathcal{D}^+$ there exists a morphism $Y \to Y'$ where $Y' \in \mathcal{D}$. We are therefore reduced to proving the following general assertion about simplicial sets:

**Lemma 6.** Let $g : Y \to W$ be a map of simplicial sets. Suppose that $|W|$ is homotopy equivalent to a space obtained from $|Y|$ by attaching finitely many cells. Then $g$ factors as a composition

$$Y \to Y' \xrightarrow{f} W$$

where $f$ is a weak homotopy equivalence and $Y'$ is obtained from $Y$ by adding finitely many simplices.

**Proof.** For simplicity, let us assume that $|W|$ is homotopy equivalent to a space obtained from $|Y|$ by attaching a single $n$-cell (the proof in the general case is similar). This $n$-cell is attached via a map $S^{n-1} \to |Y|$, which can be obtained as the geometric realization of a map of simplicial sets $A \to Y$ where $A$ is some subdivision $\partial \Delta^n$. The map $|Y| \amalg S^{n-1} D^n \to |Z|$ determines a nullhomotopy $h$ of the composite map

$$A \to Y \to Z$$

after geometric realization. Replacing $A$ by a subdivision if necessary, we may assume that the nullhomotopy $h$ arises from a nullhomotopy in the category of simplicial sets. For $n \gg 0$, we may assume that $h$ arises from a simplicial nullhomotopy of the composite map

$$A \to Y \to Z \to \text{Ex}^n Z.$$ 

We can then take

$$Y' = Y \amalg A \amalg (A \times \Delta^1) \amalg A \Delta^0.$$ 

References