Higher Torsion (Lecture 27)

November 5, 2014

Let Poly denote the ordinary category of finite polyhedra, and let $S$ denote the $\infty$-category of spaces. Over the last few lectures, we have studied the functor $K : \text{Poly} \to S$ given by

$$K(\Delta^n) = |K(\text{Shv}_{PL}^*(X \times \Delta^n))|.$$

Since every finite polyhedron has an underlying topological space, there is a forgetful functor $\iota : \text{Poly} \to S$. Let us (temporarily) use the notation $\iota_! K_\Delta$ to denote the left Kan extension of $K_\Delta$ along $\iota$. This left Kan extension can be computed in two steps:

- First, we can form the left Kan extension of $\iota$ along the forgetful functor $\text{Poly} \to \text{S}_{\text{fin}}$, where $\text{S}_{\text{fin}}$ is the $\infty$-category of finite spaces. Since $K_\Delta$ is homotopy invariant, this is equivalent to lifting $K_\Delta$ along the fully faithful embedding $\text{Fun}(\text{S}_{\text{fin}}) \to \text{Fun}(\text{Poly}, S)$. 
- We then form the left Kan extension along the fully faithful embedding $\text{S}_{\text{fin}} \to S$. This is the process of formally extending a functor $\text{S}_{\text{fin}} \to S$ to a functor $S \to S$ so that it commutes with filtered colimits.

It follows from this analysis that the restriction of $\iota_! K_\Delta$ to Poly agrees with the original functor $K_\Delta$. We will henceforth abuse notation by denoting the functor $\iota_! K_\Delta$ also by $K_\Delta$, so that we view $K_\Delta$ as a functor from spaces to spaces. The main theorem of the previous lectures gives us an explicit description of this functor: it is the domain of the assembly map in Waldhausen $A$-theory. That is, we have

$$K_\Delta(X) \simeq \Omega^\infty(X_+ \wedge A(\ast)).$$

We can use this identification to produce some $A(\ast)$-homology classes. Let $X$ be a space, and suppose we are given a finite polyhedron $Y$, a map $f : Y \to X$, and a constructible sheaf $\mathcal{F}$ on $Y$ (with values in the $\infty$-category of finite spectra). Then $\mathcal{F}$ is an object of $\text{Shv}_{PL}(Y)$ and therefore determines a point of $K(\text{Shv}_{PL}(Y))$, and therefore also of $K_\Delta(Y)$. Using the map $f$, we obtain a point of $K_\Delta(X)$ which we will denote by $\langle Y, \mathcal{F} \rangle$. In the special case where $\mathcal{F}$ is the constant sheaf on $Y$ (with value the sphere spectrum), we will denote this point simply by $\langle Y \rangle$.

We have an assembly map $K_\Delta(X) \to \Omega^\infty A(X)$. Unwinding the definitions, we see that this assembly map carries $\langle Y, \mathcal{F} \rangle$ to $[\mathcal{F}]$, where $\mathcal{F}'$ is the local system of spectra on $X$ which corepresents the functor

$$\text{Sp}^X \to \text{Sp}$$

$$\mathcal{G} \mapsto \Gamma(Y, \mathcal{F} \wedge f^* \mathcal{G})$$

(here $\Gamma$ denotes the global sections functor). In the special case where $\mathcal{F}$ is the constant sheaf, this functor is given by

$$\Gamma(Y, f^* \mathcal{G}) = \text{Map}_{\text{Sp}^X}(\Sigma Y, f^* \mathcal{G}) = \text{Map}_{\text{Sp}^X}(f_! \Sigma Y, f^* \mathcal{G}).$$

It follows that $\mathcal{F}' \simeq f_! \Sigma Y$ (where $f_!$ denotes the left adjoint to pullback on local systems), so that $[\mathcal{F}']$ can be identified with the point $[Y] \in \Omega^\infty A(X)$ studied in Lecture 21. This analysis proves the following:
**Proposition 1.** Let \( X \) be any space. For any finite polyhedron \( Y \) and any map \( f : Y \to X \), the assembly map \( K_{\Delta}(X) \to \Omega^\infty A(X) \) carries \( \langle Y \rangle \in K_{\Delta}(X) \) to \( [Y] \in \Omega^\infty A(X) \).

All of the preceding considerations can be generalized to “allow parameters”. Let us be more precise. Fix a topological space \( X \). We define Kan complexes \( M_X \) and \( M^h_X \) as follows:

- The \( n \)-simplices of \( M_X \) are finite polyhedra \( Y \subseteq \Delta^n \times \mathbb{R}^\infty \) equipped with a map \( f : Y \to X \), for which the projection \( Y \to \Delta^n \) is a PL fibration.

- The \( n \)-simplices of \( M^h_X \) are subspaces \( Y \subseteq \Delta^n \times \mathbb{R}^\infty \) equipped with a map \( f : Y \to X \) for which the projection \( Y \to \Delta^n \) is a fibration with finitely dominated fibers.

The construction \((Y \to X) \mapsto [Y]\) can be naturally refined to a map of Kan complexes \( M^h_X \to \Omega^\infty A(X) \), and the construction \((Y \to X) \mapsto \langle Y \rangle\) can be naturally refined to a map of Kan complexes \( M_X \to K_{\Delta}(X) \).

Repeating the analysis that preceded Proposition 1, we obtain the following refinement:

**Proposition 2.** Let \( X \) be any space. Then the diagram

\[
\begin{array}{ccc}
M_X & \longrightarrow & K_{\Delta}(X) \\
\downarrow & & \downarrow \\
M^h_X & \longrightarrow & \Omega^\infty A(X)
\end{array}
\]

commutes (up to canonical homotopy).

Let us now suppose that the space \( X \) itself is finitely dominated. In this case, the Kan complex \( M^h_X \) contains a contractible path component whose vertices are homotopy equivalences \( Y \to X \). Let us denote this path component by \( M^h_X \). We have a diagram of homotopy pullback squares

\[
\begin{array}{ccc}
M_X \times_{M^h_X} M^h_X & \longrightarrow & M_X \\
\downarrow & & \downarrow \\
M^h_X & \longrightarrow & M^h
\end{array}
\]

In other words, the homotopy fiber of the map \( M \to M^h \) over \( X \) can be identified with \( M_X \times M^h_X \). Applying Proposition 2, we obtain a map

\[
M \times M^h \{X\} \simeq M_X \times M^h_X \quad \cong \quad K_{\Delta}(X) \times_{\Omega A(X)} M^h_X \\
\longrightarrow \quad K_{\Delta}(X) \times_{\Omega A(X)} \{[X]\}.
\]

We can now give a more precise formulation of the main result of the second part of this course:

**Theorem 3.** Let \( X \) be a finitely dominated space. Then the map

\[
M \times M^h \{X\} \to K_{\Delta}(X) \times_{\Omega A(X)} \{[X]\}
\]

is a homotopy equivalence.

**Example 4.** Theorem 3 implies that the homotopy fiber \( M \times M^h \{X\} \) is either empty (in case \( X \) has non-vanishing Wall obstruction) or a torsor for the infinite loop space

\[
\text{fib}(K_{\Delta}(X) \to \Omega^\infty A(X)) \simeq \Omega^\infty A(X).
\]
where Wh(X) denotes the (piecewise-linear) Whitehead spectrum of X.

If X itself is given as a finite polyhedron, then the space $M \times M^b \{ X \}$ has a canonical base point. In this case, we obtain a canonical homotopy equivalence

$$\tau : M \times M^b \{ X \} \simeq \Omega^{\infty+1} \text{Wh}(X).$$

Note that the points of $M \times M^b \{ X \}$ can be identified with pairs $(Y, f)$, where $Y$ is a finite polyhedron and $f : Y \to X$ is a homotopy equivalence. If $X$ itself is a finite polyhedron, then the “identity component” of $M \times M^b \{ X \}$ consists of those pairs $(Y, f)$ where $f$ is a simple homotopy equivalence. It follows from Theorem 3 that $f$ is a simple homotopy equivalence if and only if a certain element $\tau(Y, f) \in \pi_1 \text{Wh}(X)$ vanishes. If $X$ is connected with fundamental group $G$, we have seen that there is a canonical isomorphism of $\pi_1 \text{Wh}(X)$ with the Whitehead group $\text{Wh}(G)$ of $G$, so we can identify $\tau(Y, f)$ with an element of $\text{Wh}(G)$.

**Proposition 5.** In the situation above, the element $\tau(Y, f) \in \text{Wh}(G)$ coincides with the Whitehead torsion of the homotopy equivalence $f$ (as defined in Lectures 3 and 4).

Combining Proposition 5 with Theorem 3, we obtain another proof of the main result from Lecture 4: the homotopy equivalence $f : Y \to X$ is simple if and only if its Whitehead torsion vanishes. In other words, Proposition 5 allows us to regard Theorem 3 as a generalization of the main result of Lecture 4 (and, as we have already noted, Theorem 3 also generalizes the theory of the Wall obstruction).

Let us informally sketch a proof of Proposition 5. Without loss of generality, we may assume that $Y$ and $X$ have been equipped with triangulations that are compatible with the map $f$. Assume that $X$ is connected with fundamental group $G$. We have a pair of points

$$\langle X \rangle, \langle Y \rangle \in K_\Delta(X),$$

having images $[X], [Y] \in \Omega^\infty A(X)$. Our assumption that $f$ is a homotopy equivalence supplies an equivalence of local systems $f_*S_X \simeq S_Y$, which gives a path $p$ joining $[X]$ and $[Y]$ in $\Omega^\infty A(X)$. This path gives a lift of $\langle X \rangle - \langle Y \rangle$ to the homotopy fiber

$$\Omega^{\infty+1} \text{Wh}(X) \simeq \text{fib}(K_\Delta(A) \to \Omega^\infty A(X),$$

and $\tau(Y, f)$ is the path component of this lift. Note that the map $\pi_0 K_\Delta(X) \to \pi_0 A(X)$ is injective, so that $\langle X \rangle$ and $\langle Y \rangle$ belong to the same path component of $\Omega^\infty A(X)$. If we choose a path $q$ from $\langle X \rangle$ to $\langle Y \rangle$, then we can combine $p$ with the image of $q$ to form a closed loop in the space $\Omega^\infty A(X)$. This loop determines an element $\eta \in \pi_1 A(X) \simeq K_1(\mathbb{Z}[G])$, which is preimage of $\tau(Y, f)$ under the connecting homomorphism

$$\pi_1 A(X) \to \pi_0 (\text{fib} K_\Delta(X) \to \Omega^\infty A(X)).$$

Note that the element $\eta$ depends on the choice of path $q$.

Let $\Sigma(X)$ and $\Sigma(Y)$ denote the set of simplices of $X$ and $Y$, respectively. Let $S_X$ and $S_Y$ denote the constant sheaves (with value the sphere spectrum) on $X$ and $Y$, respectively. For each simplex $\sigma$ of $X$ (or $Y$), let $S_{\sigma}$ denote the constructible sheaf on $X$ (or $Y$) taking the value $S$ on $\sigma$ and 0 elsewhere (in other words, the sheaf which is “extended by zero” from the interior of $\sigma$) and let $S_{\sigma}^n$ denote the $n$th suspension of $S_{\sigma}$. Note that

$$f_*S_{\sigma} \simeq S_{f(\sigma)}^{\dim f(\sigma) - \dim(\sigma)}.$$

For each $\sigma \in \Sigma(X) \cup \Sigma(Y)$, consider the point $e_\sigma \in K_\Delta(X)$ given by

$$e_\sigma = \begin{cases} [S_{\sigma}] & \text{if } \Sigma \in \Sigma(X) \\ -[S_{f(\sigma)}^{\dim f(\sigma) - \dim(\sigma)}] & \text{if } \Sigma \in \Sigma(Y) \end{cases}$$

Using the additivity theorem, we can choose a path from the difference $\langle X \rangle - \langle Y \rangle$ to the sum $\sum_{\sigma \in \Sigma(X) \cup \Sigma(Y)} e_\sigma$. Let $E$ denote the union of the set of even-dimensional simplices of $X$ and odd-dimensional simplices of $Y$. 

3
and let \( E' \) denote the union of the set of odd-dimensional simplices of \( X \) and even-dimensional simplices of \( Y \). Note that \( \pi_0 K_\Delta(X) \cong \mathbb{Z} \), and \( e_\sigma \) belongs to the path component \( 1 \) if \( \sigma \in E \) and the path component \(-1\) if \( \sigma \in E' \). Since \( f \) is a homotopy equivalence, \( X \) and \( Y \) have the same Euler characteristic and therefore \( E \) and \( E' \) have the same size. We may therefore choose a bijection \( \beta : E \cong E' \). For each \( \sigma \in E \), we can choose a path \( q_\sigma \) in \( K_\Delta(X) \) from \( e_\sigma + e_\beta(\sigma) \) to the base point; note that these paths are ambiguous up to an element of \( \pi_1 K_\Delta(X) \cong \mathbb{G} \oplus \mathbb{Z}/2\mathbb{Z} \). The sum of these paths determines a path \( q \) from \( \langle X \rangle - \langle Y \rangle \) to the base point.

Unwinding the definitions, we see that the image of \([X] - [Y] \) in \( K(\mathbb{Z}[G]) \) can be represented by the relative cellular chain complex \( C_*(X,Y;\mathbb{Z}[G]) \). The given triangulations of \( X \) and \( Y \) determine a basis for \( C_*(X,Y;\mathbb{Z}[G]) \) as a \( \mathbb{Z}[G] \)-module, where the basis elements are ambiguous up to \( \pm \mathbb{G} \). We have two paths from \([C_*(X,Y;\mathbb{Z}[G])]\) to the base point of \( K(\mathbb{Z}[G]) \), given as follows:

(a) The image of \( p \) determines a path from \([C_*(X,Y;\mathbb{Z}[G])]\) to the base point of \( K(\mathbb{Z}[G]) \) which arises from the observation that \( C_*(X,Y;\mathbb{Z}[G]) \) is an acyclic complex (because \( f \) is a homotopy equivalence), and therefore represents a zero object of the \( \infty \)-category \( \text{Rep}_{\mathbb{Z}[G]} \).

(b) The image of the path \( q \) determines a path from \([C_*(X,Y;\mathbb{Z}[G])]\) to the base point of \( K(\mathbb{Z}[G]) \). After possibly modifying our choice of basis, we can arrange that this path is obtained by first invoking the additivity theorem to construct a path from \([C_*(X,Y;\mathbb{Z}[G])]\) to the point represented by the sum

\[
\bigoplus_{\sigma \in \Sigma(X)} [\Sigma^{\dim(\sigma)} \mathbb{Z}[G]] + \bigoplus_{\sigma \in \Sigma(Y)} [\Sigma^{\dim(\sigma)+1} \mathbb{Z}[G]],
\]

and then connecting this latter sum to the base point by matching factors using the bijection \( \beta \).

We are therefore reduced to the following statement, which we leave as a (tedious) exercise:

**Exercise 6.** Let \( R \) be a ring and let \( F_* \) be a bounded acyclic chain complex of free \( R \)-modules, where \( \chi(F_*) = 0 \) (the latter condition is automatic if \( R \) has the form \( \mathbb{Z}[G] \)). Suppose we have chosen a basis \( \{e_i, e'_i\} \) for \( F_* \), where each \( e_i \) is homogeneous of even degree \( d_i \), and each \( e'_i \) is homogeneous of odd degree \( d'_i \). Then the torsion \( \tau(F_*) \in K_1(R) \cong \pi_1 K(R) \) (as defined in Lecture 3) can be represented as the “difference” between two paths from \([F_*]\) to the base point of the space \( K(R) \):

(a) The path obtained from the observation that the chain complex \( F_* \) represents the zero object of \( \text{Mod}_R \) (since \( F_* \) is acyclic).

(b) The path obtained by first using the additivity theorem to construct a path from \([F_*]\) to the sum \( \sum d_i [\Sigma^d_i R] \oplus [\Sigma^{d'_i} R] \), then connecting each \([\Sigma^d_i R] + [\Sigma^{d'_i} R] \) to the base point using the fact that \( d_i \) and \( d'_i \) have different parities.

**References**
