Our goal in this lecture is to complete the proof of the following result:

**Proposition 1.** Suppose we are given a pushout diagram

\[
\begin{array}{c}
X_01 \longrightarrow X_0 \\
\downarrow \quad \downarrow \\
X_1 \longrightarrow X
\end{array}
\]

in the $\infty$-category of finite spaces. Then the diagram

\[
\begin{array}{c}
K_\Delta(X_{01}) \longrightarrow K_\Delta(X_0) \\
\downarrow \quad \downarrow \\
K_\Delta(X_1) \longrightarrow K_\Delta(X)
\end{array}
\]

is also a pushout square of $E_\infty$-spaces.

To prove Proposition 1, it will be convenient to consider instead the diagram of connected deloopings

\[
\begin{array}{c}
\Omega^{-1}K_\Delta(X_{01}) \longrightarrow \Omega^{-1}K_\Delta(X_0) \\
\downarrow \quad \downarrow \\
\Omega^{-1}K_\Delta(X_1) \longrightarrow \Omega^{-1}K_\Delta(X)
\end{array}
\]

We wish to compare $\Omega^{-1}K_\Delta(X)$ with the homotopy pushout of the rest of the diagram, which is computed as the geometric realization of a simplicial space

\[
\text{Bar}_\bullet(\Omega^{-1}K_\Delta(X_0), \Omega^{-1}K_\Delta(X_{01}), \Omega^{-1}K_\Delta(X_1))
\]

given by

\[
\text{Bar}_n(\Omega^{-1}K_\Delta(X_0), \Omega^{-1}K_\Delta(X_{01}), \Omega^{-1}K_\Delta(X_1)) = (\Omega^{-1}K_\Delta(X_0)) \times (\Omega^{-1}K_\Delta(X_{01}))^n \times (\Omega^{-1}K_\Delta(X_1)).
\]

As in the previous lecture, let us assume that $X_0$ and $X_1$ are finite polyhedra, that $X_{01}$ is a subpolyhedron of each, and that $X$ is given by the two-sided mapping cylinder

\[
X_0 \amalg_{\{0\}} \amalg_{\{1\}} X_{01} \amalg_{\{0\}} X_1.
\]

For each $t \in (0, 1)$, let $X_t$ denote the subpolyhedron $\{t\} \times X_{01} \subseteq X$. Recall that a sheaf $\mathcal{F} \in \text{Shv}_{PL}(X \times S)$ is **transverse to $t$** if $\mathcal{F}|_{X_t \times S}$ is ULA over $S$. More generally, if $T$ is a finite subset of the open interval $(0, 1)$, we
will say that \( F \in \text{Shv}_{PL}^{S,T}(X \times S) \) is transverse to \( T \) if it is transverse to \( t \) for each \( t \in T \). Let \( \text{Shv}_{PL}^{S,T}(X \times S) \) denote the full subcategory of \( \text{Shv}_{PL}^{S}(X \times S) \) spanned by those sheaves which are transverse to \( T \), and let \( K^{\Delta}_{\Delta}(X) \) denote the geometric realization of the simplicial space given by

\[
K(\text{Shv}_{PL}^{S,T}(X \times \Delta^{\bullet})).
\]

We proved in the last lecture that if \( T \) has a single element, then \( K^{T}_{\Delta}(X) \) can be identified with the product

\[
K_{\Delta}(X_{0}) \times K_{\Delta}(X_{01}) \times K_{\Delta}(X_{1}).
\]

One can carry out a similar argument for larger finite sets \( T = \{ t_{1}, \ldots, t_{n} \} \): every object of \( \text{Shv}_{PL}^{S,T}(X \times S) \) has a canonical filtration whose subquotients are sheaves which are supported either on \( X_{t_{i}} \times S \) for \( 1 \leq i \leq n \), on \( X_{<t_{i}} \times S \) (which contains \( X_{0} \times S \) as a deformation retract), or on \( X_{>t_{n}} \times S \) (which contains \( X_{1} \times S \) as a deformation retract), or on \( X_{01} \times \{ t_{i}, t_{i+1} \} \times S \) for \( 1 \leq i < n \). This analysis supplies a homotopy equivalence

\[
K^{T}_{\Delta}(X) \simeq K_{\Delta}(X_{0}) \times K_{\Delta}(X_{01})^{2n-1} \times K_{\Delta}(X_{1}).
\]

We can be more precise: the construction \( T \mapsto K^{T}_{\Delta}(X) \) is contravariantly functorial in \( T \), and there is a functorial identification

\[
\Omega^{-1}K^{T}_{\Delta}(X) \simeq \text{Bar}_{\alpha(T)}(\Omega^{-1}K_{\Delta}(X_{0}), \Omega^{-1}K_{\Delta}(X_{01}), \Omega^{-1}K_{\Delta}(X_{1}))
\]

where \( \alpha(T) \) is the finite linearly ordered set given by \( T \times \{ 0, 1 \} \), equipped with the lexicographical ordering.

**Lemma 2.** Let \( P \) be the poset of nonempty finite subsets of \( \{ 0, 1 \} \) and let \( \Delta \) be the category of finite nonempty linearly ordered sets. Then the construction

\[
T \mapsto T \times \{ 0, 1 \}
\]

determines a right cofinal functor

\[
\alpha : P \to \Delta.
\]

**Proof.** Fix an object \( [n] = \{ 0 < 1 < \cdots < n \} \in \Delta; \) we wish to show that the category (in fact, poset) \( Q = P \times_{\Delta} \Delta_{[n]} \) is weakly contractible. For each \( \epsilon > 0 \), let \( P_{\epsilon} \subseteq P \) denote the collection of all nonempty finite subsets of \( (\epsilon, 1) \), and let \( Q_{\epsilon} \) denote the inverse image in \( Q \) of \( P_{\epsilon} \). Then \( Q \) can be written as a directed union of the posets \( Q_{\epsilon} \). Consequently, to show that \( Q \) is weakly contractible, it will suffice to show that each of the inclusion maps \( Q_{\epsilon} \hookrightarrow Q \) is nullhomotopic. We can identify the elements of \( Q \) with pairs \( (T, f) \) where \( T \subseteq \{ 0, 1 \} \) is a nonempty finite set and \( f : T \times \{ 0, 1 \} \to \{ n \} \) is a monotone map. If \( (T, f) \in Q_{\epsilon} \), then we have natural maps

\[
(T, f) \mapsto (T \cup \{ \epsilon \}, f_{+}) \mapsto (\{ \epsilon \}, g),
\]

where \( g : \{ \epsilon \} \times \{ 0, 1 \} \to \{ n \} \) is the constant map taking the value 0 and \( f_{+} \) is the amalgamation of the maps \( f \) and \( g \). These maps determine a homotopy from the inclusion \( Q_{\epsilon} \hookrightarrow Q \) to the constant map \( Q_{\epsilon} \to \{ (\{ \epsilon \}, g) \} \subseteq Q \). \( \square \)

It follows from Lemma 2 that the pushout

\[
\Omega^{-1}K_{\Delta}(X_{0}) \amalg_{\Omega^{-1}K_{\Delta}(X_{01})} \Omega^{-1}K_{\Delta}(X_{1})
\]

(formed in the \( \infty \)-category of \( E_{\infty} \)-spaces) can also be written as

\[
\lim_{T \subseteq \{ 0, 1 \}} \Omega^{-1}K^{T}_{\Delta}(X)
\]

(formed in the \( \infty \)-category of spaces). We may therefore rewrite Proposition 1 as follows:
Proposition 3. The canonical map
\[ \lim_{T \subseteq (0,1)} \Omega^{-1} K^T_{\Delta}(X) \to \Omega^{-1} K_{\Delta}(X) \]
is a homotopy equivalence of spaces.

Note that the map of Proposition 3 can be obtained as the geometric realization of a map of bisimplicial spaces which is given (in bidegree \((m, n)\)) by
\[ \theta_m : \lim_{T \subseteq (0,1)} S_n \text{Shv}^m_{PL}(X \times \Delta^m) \to S_n \text{Shv}^m_{PL}(X \times \Delta^m). \]

Let us regard \(n\) as fixed for the remainder of this lecture. We will prove Proposition 3 by showing that the map of geometric realizations
\[ | \lim_{T \subseteq (0,1)} S_n \text{Shv}^\bullet_{PL}(X \times \Delta^\bullet) | \to | S_n \text{Shv}^\bullet_{PL}(X \times \Delta^\bullet) | \]
is a homotopy equivalence. Note that for fixed \(m\) and \(T \subseteq (0,1)\), the map
\[ S_n \text{Shv}^m_{PL}(X \times \Delta^m) \to S_n \text{Shv}^m_{PL}(X \times \Delta^m) \]
is the inclusion of a summand: the target space classifies diagrams
\[ \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \]
in \(\text{Shv}^m_{PL}(X \times \Delta^m)\), and the domain classifies diagrams which have the additional property that each \(\mathcal{F}_i\) is transverse to each \(t \in T\). If the diagram \(\mathcal{F}_1 \to \cdots \to \mathcal{F}_n\) is fixed, then the collection of those \(t \in (0, 1)\) such that each \(\mathcal{F}_i\) is transverse to \(T\) comprise a subset \(U \subseteq (0,1)\), and the homotopy fiber of the map \(\theta_m\) over the point corresponding to \(\mathcal{F}_1 \to \cdots \to \mathcal{F}_n\) can be identified with the nerve of the category of nonempty finite subsets of \(U\). This category is either empty (if \(U = \emptyset\)) or contractible (if \(U \neq \emptyset\)). It follows that each of the maps \(\theta_m\) exhibits \(\lim_{T \subseteq (0,1)} S_n \text{Shv}^\bullet_{PL}(X \times \Delta^\bullet)\) as a summand of \(S_n \text{Shv}^m_{PL}(X \times \Delta^m)\): namely, the summand consisting of those diagrams \(\mathcal{F}_1 \to \cdots \to \mathcal{F}_n\) which are all transverse to some \(t \in (0, 1)\).

Construction 4. Let \(Z_\bullet\) be a simplicial space. For every simplicial set \(K\), we let \(Z_\bullet(K)\) denote the homotopy inverse limit \(\lim_{-\Delta^q} Z_\bullet\).

We define a new simplicial space \(\text{Ex}(Z_\bullet)\) by the formula
\[ \text{Ex}(Z_\bullet)_q = Z_\bullet(\text{Sd} \Delta^q). \]

Note that the “last vertex maps” \(\text{Sd} \Delta^q \to \Delta^q\) are compatible as \(q\) varies and give rise to a map of simplicial spaces \(\rho_{Z_\bullet} : Z_\bullet \to \text{Ex}(Z_\bullet)\).

Lemma 5. For any simplicial space \(Z_\bullet\), the map \(\rho_{Z_\bullet} : Z_\bullet \to \text{Ex}(Z_\bullet)\) induces a homotopy equivalence of geometric realizations
\[ |Z_\bullet| \to |\text{Ex}(Z_\bullet)|. \]

Sketch. If \(Z_\bullet\) is a simplicial set (meaning that each \(Z_n\) is discrete), then this is classical. The general case can be reduced to this: the \(\infty\)-category of simplicial spaces is the underlying \(\infty\)-category of the model category of bisimplicial sets, equipped with the Reedy model structure. Moreover, if \(Z_{\bullet\bullet}\) is a bisimplicial set which is Reedy fibrant, then the \(\text{Ex}\) of Construction 4 can be computed by levelwise application of the usual \(\text{Ex}\) functor on simplicial sets.
Let us now specialize to the case where \( Z_\bullet \) is the simplicial space \( S_n \text{Shv}_{PL}^\Delta (X \times | \Delta^m |) \). In this case, we can identify \( \text{Ex}(Z_\bullet) \) with the simplicial space whose \( m \)-simplices are given by \( S_n \text{Shv}_{PL}^{\Delta m} | \Delta^m | \). Recall that there is a canonical piecewise linear homeomorphism of \( | \Delta^m | \) with \( | \Delta^m | \), which is functorial for injective maps between simplices. These homeomorphisms determine an equivalence \( \theta Z_\bullet \) between the underlying semisimplicial spaces of \( Z_\bullet \) and \( \text{Ex}(Z_\bullet) \), which we will denote by \( Z_\bullet^\ast \) and \( \text{Ex}(Z_\bullet)^\ast \).

Let \( Y_\bullet \) denote the simplicial subspace of \( Z_\bullet \) given by \( \lim_{\to T \subseteq (0,1)} S_n \text{Shv}_{PL}^{\Delta m, T} (X \times | \Delta^m |) \). The map \( \theta Z_\bullet \) restricts to a morphism of underlying semisimplicial spaces
\[
\theta Y_\bullet : Y_\bullet^\ast \to \text{Ex}(Y_\bullet)^\ast
\]
(which is now not a levelwise homotopy equivalence). This is not identical to the map of semisimplicial spaces \( \rho Y_\bullet \) appearing in Lemma 5. However, they differ by a simplicial homotopy and therefore induce the same map after passing to geometric realizations. It follows from Lemma 5 that \( \theta Y_\bullet \) induces a homotopy equivalence
\[
| Y_\bullet^\ast | \to | \text{Ex}(Y_\bullet)^\ast |.
\]

We wish to show that the inclusion \( i_\bullet : Y_\bullet \to Z_\bullet \) induces a homotopy equivalence of geometric realizations. For each integer \( p \geq 0 \), let \( \text{Ex}^p(Y_\bullet) \) denote the result of \( p \)-fold application of the functor \( \text{Ex} \) to the simplicial space \( Y_\bullet \), and define \( \text{Ex}^p(Z_\bullet) \) similarly. Using the above arguments we can identify \( | i_\bullet | \) with the induced of geometric realizations
\[
| \lim_{\to} \text{Ex}^p(Y_\bullet)^\ast | \to | \lim_{\to} \text{Ex}^p(Z_\bullet)^\ast |.
\]

It will therefore suffice to prove the following:

**Proposition 6.** For each integer \( m \), the canonical map \( \lim_{\to} \text{Ex}^p(Y_\bullet)_m \to \lim_{\to} \text{Ex}^p(Z_\bullet)_m \) is a homotopy equivalence.

Unwinding the definitions, Proposition 6 asserts that for every diagram \( \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \) in \( \text{Shv}_{PL}^{\Delta m} (X \times | \Delta^m |) \), there exists an integer \( p \geq 0 \) such that on each simplex \( \sigma \) of the \( p \)-fold barycentric subdivision of \( | \Delta^m | \), there exists a real number \( t_\sigma \) which is transverse to each \( \mathcal{F}_i | X \times \sigma \).

Choose compatible triangulations \( \Sigma \) of \( X_{01} \times [0,1] \times | \Delta^m | \), \( \Sigma' \) of \( [0,1] \times | \Delta^m | \), and \( \Sigma'' \) of \( | \Delta^m | \), such that each \( \Sigma_i \) is constructible with respect to \( \Sigma \). Let \( V \subseteq [0,1] \times | \Delta^m | \) be the union of the interiors of those simplices \( \sigma \in \Sigma' \) for which the map \( \sigma \to | \Delta^m | \) is not injective. It is easy to see that \( V \subseteq (0,1) \times | \Delta^m | \) and that the projection map \( V \to | \Delta^m | \) is surjective. Since \( V \) is open, there exists an open cover \( U_{\alpha} \) of \( | \Delta^m | \) and a collection of real numbers \( t_\alpha \in (0,1) \) such that each product \( \{ t_\alpha \} \times U_{\alpha} \) is contained in \( V \). Choose \( p \geq 0 \) so that each simplex \( \sigma \) of the \( p \)-fold barycentric subdivision of \( | \Delta^m | \) is contained in some \( U_{\alpha} \). We claim that \( p \) satisfies our requirements: more precisely, each restriction \( \mathcal{F}_i | X \times \sigma \) is transverse to \( t_\alpha \). To prove this, it will suffice to show that \( \mathcal{F}_i | X_{01} \times [0,1] \times | \Delta^m | \) is ULA over \( [0,1] \times | \Delta^m | \) over the open set \( V \).

Let
\[
f : X_{01} \times [0,1] \times | \Delta^m | \to [0,1] \times | \Delta^m |
\]
and
\[
g : [0,1] \times | \Delta^m | \to | \Delta^m |
\]
be the projection maps, let \( \theta_0 \in \Sigma \), let \( \sigma_0 = f(\theta_0) \in \Sigma' \), and let \( \sigma \in \Sigma' \) be a simplex containing \( \sigma_0 \). Set \( \tau_0 = g(\sigma_0) \in \Sigma'' \) and \( \tau = g(\sigma) \in \Sigma'' \). We wish to show that if the interior of \( \sigma_0 \) is contained in \( V \), then the canonical map
\[
\mathcal{F}_i (\theta_0) \to \lim_{\theta_0 \subseteq \theta, f(\theta) = \sigma} \mathcal{F}_i (\theta)
\]
and
\[
\mathcal{F}_i (\theta_0) \to \lim_{\theta_0 \subseteq \theta, (g \circ f)(\theta) = \tau} \mathcal{F}_i (\theta)
\]
is an equivalence. Our assumption that the interior of \( \sigma_0 \) is contained in \( V \) guarantees that \( \sigma \) is the unique simplex of \( \Sigma' \) which contains \( \sigma_0 \) and whose image is \( \tau \), so we can rewrite our map as
\[
\mathcal{F}_i (\theta_0) \to \lim_{\theta_0 \subseteq \theta, (g \circ f)(\theta) = \tau} \mathcal{F}_i (\theta).
\]
This map is an equivalence by virtue of our assumption that \( \mathcal{F}_i \) is ULA over \( | \Delta^m | \).