Let $X$ be a finite polyhedron. In the previous lecture, we introduced an infinite loop space $K_{\Delta}(X)$, which is given by the geometric realization of the simplicial space

$$[n] \mapsto K(\text{Shv}_{PL}^\Delta(X \times \Delta^n)).$$

Moreover, we constructed a map of connective spectra

$$\Omega^{-\infty}K_{\Delta}(X) \to A(X).$$

Our goal in this lecture and the next is to prove the following:

**Theorem 1.** The map $K_{\Delta}(X) \to \Omega^\infty A(X)$ is a model for the assembly map in $A$-theory. In particular, there is a canonical homotopy equivalence $K_{\Delta}(X) \simeq \Omega^\infty(X_+ \wedge A(\ast))$.

As a first step, we consider functoriality in $X$. Recall that for any map of finite polyhedra $f : X \to Y$, the pushforward map

$$(f \times \text{id})_* : \text{Shv}(X \times S) \to \text{Shv}(Y \times S)$$

preserves the property of being ULA over $S$. It follows that we obtain a map of simplicial $\infty$-categories

$$\text{Shv}_{PL}^\Delta(X \times \Delta^\bullet) \to \text{Shv}_{PL}^\Delta(Y \times \Delta^\bullet).$$

Taking $K$-theory and passing to geometric realizations, we obtain a map of infinite loop spaces $K_{\Delta}(X) \to K_{\Delta}(Y)$. In other words, we can regard $K_{\Delta}$ as a functor from the ordinary category Poly of finite polyhedra (with morphisms given by PL maps) to the $\infty$-category of spaces (or even of $E_\infty$-spaces).

The category Poly has a canonical simplicial enrichment: to every pair of finite polyhedra $X$ and $Y$, we can associate a Kan complex $\text{Map}(X,Y)$ whose $n$-simplices are given by piecewise linear maps from $X \times \Delta^n$ into $Y$. As a simplicially enriched category, Poly is weakly equivalent to the simplicially enriched category of finite CW complexes. This follows from two observations:

- Every finite CW complex is homotopy equivalent to a finite polyhedron.
- For every pair of finite polyhedra $X$ and $Y$, the Kan complex $\text{Map}(X,Y)$ is homotopy equivalent to the singular simplicial set of the topological space $Y^X$ of all continuous maps from $X$ into $Y$ (more informally: there are no obstructions to approximating arbitrary continuous maps between polyhedra by piecewise-linear maps).

Using this fact, it is not difficult to show that the $\infty$-category $S^{\text{fin}}$ of finite spaces can be obtained from the ordinary category Poly by formally inverting all maps of the form $X \times \Delta^n \to X$. In fact, it suffices to consider the case $n = 1$ (since the $n$-simplex $\Delta^n$ is a retract of a product of copies of $\Delta^1$). In other words, we have the following:
Claim 2. Let $\mathcal{C}$ be an $\infty$-category. Then composition with the canonical map $\text{Poly} \to \mathcal{S}^{\text{fin}}$ induces a fully faithful embedding

$$\text{Fun}(\mathcal{S}^{\text{fin}}, \mathcal{C}) \to \text{Fun}(\text{Poly}, \mathcal{C}),$$

whose essential image is spanned by the collection of those functors $F : \text{Poly} \to \mathcal{C}$ with the property that for any finite polyhedron $X$, the induced map $F(X \times \Delta^1) \to F(X)$ is an equivalence in $\mathcal{C}$.

We would like to apply Claim 2 to the functor $X \mapsto K_\Delta(X)$.

Proposition 3. For any finite polyhedron $X$, the canonical map $K_\Delta(X \times \Delta^1) \to K_\Delta(X)$ is a homotopy equivalence.

The map of Proposition 3 has a right homotopy inverse, induced by the inclusion $X \times \{0\} \hookrightarrow X \times \Delta^1$. To check that this map is a left homotopy inverse, it will suffice to establish the following:

Lemma 4. Let $f, g : X \to Y$ be homotopic maps of finite polyhedra. Then $f$ and $g$ induce homotopic maps $f_*, g_* : K_\Delta(X) \to K_\Delta(Y)$.

Proof. It suffices to prove Lemma 4 in the “universal” case where $Y = X \times \Delta^1$ and $f$ and $g$ are the two inclusions $X \times \{1\} \hookrightarrow X \times \Delta^1$. Let $\mathcal{J}$ denote the slice category $\Delta/\{1\}$ of nonempty finite linearly ordered sets $[n]$ equipped with a map $[n] \to [1]$. Then $\mathcal{J}$ contains full subcategories $\mathcal{J}_0, \mathcal{J}_1 \subseteq \mathcal{J}$, spanned by those objects of the form $[n] \to \{0\} \subseteq [1]$ and $[n] \to \{1\} \subseteq [1]$, respectively. Each of these subcategories is equivalent to $\Delta$. To each object $[n] \to [1]$ in $\mathcal{J}$, we can associate a map of finite polyhedra

$$X \times \Delta^n \to X \times \Delta^1 \times \Delta^n,$$

which induces a pushforward functor

$$\text{Shv}^\Delta_\text{PL}(X \times \Delta^n) \to \text{Shv}^\Delta_\text{PL}(X \times \Delta^1 \times \Delta^n).$$

Taking $K$-theory and passing to the colimit, we obtain a map

$$\lim_{[n] \in \mathcal{J}} K(\text{Shv}^\Delta_\text{PL}(X \times \Delta^n)) \to K_\Delta(X \times \Delta^1).$$

Note that the composition of this map with the natural maps

$$\lim_{[n] \in \mathcal{J}_0} K(\text{Shv}^\Delta_\text{PL}(X \times \Delta^n)) \to \lim_{[n] \in \mathcal{J}} K(\text{Shv}^\Delta_\text{PL}(X \times \Delta^n))$$

$$\lim_{[n] \in \mathcal{J}_1} K(\text{Shv}^\Delta_\text{PL}(X \times \Delta^n)) \to \lim_{[n] \in \mathcal{J}} K(\text{Shv}^\Delta_\text{PL}(X \times \Delta^n))$$

coincide with $f_*$ and $g_*$, respectively. It will therefore suffice to show that these latter maps are homotopy equivalences. Both have left homotopy inverses induced by the map

$$\lim_{[n] \in \mathcal{J}} K(\text{Shv}^\Delta_\text{PL}(X \times \Delta^n)) \to K_\Delta(X)$$

determined by the forgetful functor $\pi : \mathcal{J} \to \Delta$. To show that this map is a homotopy equivalence, it will suffice to show that $\pi$ is right cofinal. In other words, it will suffice to show that for each object $[n] \in \Delta$, the category

$$\mathcal{J} \times \Delta/\{n\} = \Delta/\{1\} \times \Delta/\{n\}$$

is weakly contractible. This is clear, since it is the category of simplices of the weakly contractible simplicial set $\Delta^1 \times \Delta^n$. 

\[\square\]
It follows from Proposition 3 and Claim 2 that we can regard the construction $X \mapsto K_\Delta(X)$ as a functor from the $\infty$-category $\text{S}^\text{fin}$ of finite spaces to the $\infty$-category of $E_\infty$ spaces. Moreover, since the map

$$\Omega^{-\infty}K_\Delta(X) \to A(X)$$

called the previous lecture was functorial for maps of finite polyhedra, it can be regarded as a natural transformation between functors from $\text{S}^{\text{fin}}$ to spectra. We next make a simple observation:

**Proposition 5.** The map $\Omega^{-\infty}K_\Delta(X) \to A(X)$ is an equivalence when $X$ is a point.

**Proof.** When $X$ is a point, a constructible sheaf $F$ on $X \times \Delta^n$ is ULA over $\Delta^n$ if and only if it is locally constant. Consequently, we can identify $\text{Shv}_{PL}^{\Delta^n}(X \times \Delta^n)$ with the constant simplicial $\infty$-category taking the value $\text{Sp}^{\text{fin}}$. It follows that $K_\Delta(X)$ can be identified with $K(\text{Sp}^{\text{fin}}) \simeq \Omega^{-\infty}A(X)$. □

It follows from Proposition 5 that the colimit-preserving approximations to the functors $\Omega^{-\infty}K_\Delta(X)$ and $A(X)$ are the same. In other words, for any finite space $X$ we have a commutative diagram

$$\begin{array}{ccc}
X_+ \land A(*) & \xrightarrow{\theta_X} & A(X) \\
\downarrow & & \downarrow \\
\Omega^{-\infty}K_\Delta(X) & \to & A(X).
\end{array}$$

We can now formulate a more precise version of Theorem 1: the map $\theta_X$ is a homotopy equivalence of spectra. To prove this, we note that the collection of those spaces $X$ for which $\theta_X$ is a homotopy equivalence contains the one-point space (by Proposition 5) and the empty space (since the domain and codomain of $\theta_X$ both vanish in this case). Consequently, to show that it contains all finite spaces, it will suffice to show that it is closed under (homotopy) pushouts. Since the functor $X \mapsto X_+ \land A(*)$ preserves pushout squares, we are reduced to the following:

**Proposition 6.** Suppose we are given a pushout diagram

$$\begin{array}{ccc}
X_0 & \rightarrow & X_1 \\
\downarrow & & \downarrow \\
X_01 & \rightarrow & X
\end{array}$$

in the $\infty$-category of finite spaces. Then the diagram of spectra

$$\begin{array}{ccc}
\Omega^{-\infty}K_\Delta(X_01) & \rightarrow & \Omega^{-\infty}K_\Delta(X_0) \\
\downarrow & & \downarrow \\
\Omega^{-\infty}K_\Delta(X_1) & \rightarrow & \Omega^{-\infty}K(X)
\end{array}$$

is also a homotopy pushout square.

In the statement of Proposition 6, we may assume without loss of generality that $X_01$, $X_0$, and $X_1$ are represented by finite polyhedra, and that the maps $X_01 \rightarrow X_0$ and $X_01 \rightarrow X_1$ are given by embeddings of finite polyhedra. We will assume that $X$ is given by the homotopy pushout

$$X_0 \amalg_{X_0 \times \{0\}} (X_01 \times [0,1]) \amalg_{X_0 \times \{1\}} X_1.$$ 

For each real number $t$ with $0 < t < 1$, let $X_t$ denote the subcomplex of $X$ given by $X_01 \times \{t\}$. We will say that a sheaf $\mathcal{F} \in \text{Shv}_{PL}^{S}(X \times S)$ is transverse to $X_t$ if the restriction $\mathcal{F}|_{X_t \times S}$ is ULA over $S$. Let $\text{Shv}_{PL}^{S,t}(X \times S)$ denote the full subcategory of $\text{Shv}_{PL}^{S}(X \times S)$ spanned by those sheaves which are transverse.
to $t$. As $S$ ranges over all simplices, we obtain a simplicial $\infty$-category $\text{Shv}_{PL}^\Delta(X \times \Delta^\bullet)$. Let $K^t_\Delta(X)$ denote the geometric realization of the simplicial space

$$K(\text{Shv}_{PL}^\Delta(X \times \Delta^\bullet)).$$

Let us compute $K^t_\Delta(X)$.

Let $S$ be a finite polyhedron, and let $\mathcal{C}^-_S$ and $\mathcal{C}^+_S$ denote the full subcategories of $\text{Shv}_{PL}(X \times S)$ spanned by those sheaves which are ULA over $S$ and which vanish when restricted to $X_t$. Note that $X_{\leq t}$ and $X_{\geq t}$ contain $X_0$ and $X_1$ as deformation retracts. Using a slight variant of the proof of Lemma 4, one can show that the canonical maps

$$K_\Delta(X_0) \to |\mathcal{C}^-_S|$$

$$K_\Delta(X_1) \to |\mathcal{C}^+_S|$$

are homotopy equivalences.

Note that if $\mathcal{F} \in \text{Shv}_{S,T}^S(X \times S)$, then we have a canonical fiber sequence

$$\mathcal{F}' \to \mathcal{F} \to i^* \mathcal{F}.$$

Since $\mathcal{F}$ and $i^* \mathcal{F}$ are ULA over $S$, it follows that $i_* i^* \mathcal{F}$ and $\mathcal{F}'$ are ULA over $S$. Because $\mathcal{F}'|_{X_t}$ vanishes, we can write $\mathcal{F}'$ as a direct sum $\mathcal{F}'_- \oplus \mathcal{F}'_+$, where $\mathcal{F}'_- \in \mathcal{C}^-_S$ and $\mathcal{F}'_+ \in \mathcal{C}^+_S$. The construction $\mathcal{F} \mapsto (\mathcal{F}'_-, \mathcal{F}'_+, i^* \mathcal{F})$ determines an exact functor

$$\text{Shv}_{S,T}^S(X \times S) \to \mathcal{C}^-_S \times \mathcal{C}^+_S \times \text{Shv}_{PL}^S(X_t \times S).$$

This map has a right homotopy inverse (given by pushing forward to $X \times S$ and forming the direct sum). It follows from the additivity theorem that this right homotopy inverse is actually a two-sided homotopy inverse after passing to $K$-theory. In particular, we obtain a homotopy equivalence

$$K(\text{Shv}_{S,T}^S(X \times S)) \simeq K(\mathcal{C}^-_S) \times K(\mathcal{C}^+_S) \times K(\text{Shv}_{PL}^S(X_t \times S)).$$

Taking $S$ to range over simplices and passing to geometric realizations, we obtain an equivalence

$$K^t_\Delta(X) \simeq K_\Delta(X_0) \times K_\Delta(X_1) \times K_\Delta(X_t).$$

We will elaborate more on this in the next lecture.