Additive K-Theory (Lecture 18)

October 15, 2014

Let \( \mathcal{C} \) be a pointed \( \infty \)-category which admits finite colimits, let \( \mathcal{C}_0 \subseteq \mathcal{C} \) be a full subcategory which is closed under finite colimits, and assume that every object of \( \mathcal{C} \) can be obtained as a retract of an object of \( \mathcal{C}_0 \). In Lectures 14 and 15, we saw that there can be a big difference between \( K_0(\mathcal{C}) \) and \( K_0(\mathcal{C}_0) \): in the stable case, an object \( C \in \mathcal{C} \) belongs to (the essential image of) \( \mathcal{C}_0 \) if and only if the class \( [C] \in K_0(\mathcal{C}) \) belongs to the image of the map \( K_0(\mathcal{C}_0) \hookrightarrow K_0(\mathcal{C}) \). Our first goal in this lecture is to show that the difference between \( \mathcal{C} \) and \( \mathcal{C}_0 \) disappears when we look at higher \( K \)-groups. More precisely, we have the following result:

**Proposition 1.** Let \( \mathcal{C} \) and \( \mathcal{C}_0 \) be as above. Then the canonical map \( K_0(\mathcal{C}_0) \rightarrow K_0(\mathcal{C}) \) is an isomorphism for \( n > 0 \). In other words, the diagram of spaces

\[
\begin{array}{ccc}
K(\mathcal{C}_0) & \rightarrow & K(\mathcal{C}) \\
\downarrow & & \downarrow \\
K_0(\mathcal{C}_0) & \rightarrow & K_0(\mathcal{C})
\end{array}
\]

is a homotopy pullback square.

To prove Proposition 1, we may assume without loss of generality that \( \mathcal{C} \) and \( \mathcal{C}_0 \) are stable (since the construction \( \mathcal{C} \mapsto \text{SW}(\mathcal{C}) \) has no effect on \( K \)-theory). For every group \( G \), let \( B\bullet(G) \) denote the simplicial set which models the classifying space of \( G \), so that the set \( B^n(G) \) of \( n \)-simplices of \( B\bullet(G) \) can be identified with \( G^n \). Let us describe this in a way that makes the simplicial structure of \( B\bullet(G) \) more apparent. Using additive notation for the group structure on \( G \)(we will ultimately be interested in the case where \( G \) is abelian), we can identify \( B^n(\mathcal{C}) \) with the set of maps \( f : [n]^{(2)} = \{(i,j) \in [n] \times [n] : i \leq j \} \rightarrow G \) which have the property that \( f(i,i) = 0 \) and \( f(i,j) + f(j,k) = f(i,k) \) for \( i \leq j \leq k \).

If \( \mathcal{C} \) is a pointed \( \infty \)-category which admits finite colimits, then every \( [n] \)-gapped object \( X : [n]^{(2)} \rightarrow \mathcal{C} \) determines a map \( f : [n]^{(2)} \rightarrow K_0(\mathcal{C}) \), given by \( f(i,j) = [X(i,j)] \), and \( f \) will satisfy the condition above. This construction is functorial in \( [n] \) and therefore gives rise to a map of simplicial spaces

\[
S\bullet(\mathcal{C}) \rightarrow B\bullet(K_0(\mathcal{C})).
\]

The natural map \( K(\mathcal{C}) \rightarrow K_0(\mathcal{C}) \) can then be obtained by passing to classifying spaces and then applying \( \Omega \). We may therefore rephrase Proposition 1 as follows:

**Proposition 2.** Let \( \mathcal{C} \) be a stable \( \infty \)-category and let \( \mathcal{C}_0 \) be a full stable subcategory such that every object of \( \mathcal{C} \) is a direct summand of an object of \( \mathcal{C}_0 \). Then the diagram

\[
\begin{array}{ccc}
|S\bullet(\mathcal{C}_0)| & \rightarrow & |S\bullet(\mathcal{C})| \\
\downarrow & & \downarrow \\
|B\bullet K_0(\mathcal{C}_0)| & \rightarrow & |B\bullet K_0(\mathcal{C})|
\end{array}
\]

is a homotopy pullback square.
Because the map $K_0(\mathcal{C}_0) \to K_0(\mathfrak{C})$ is injective and an object $C \in \mathfrak{C}$ belongs to $\mathcal{C}_0$ if and only if its $K$-theory class $[C]$ lifts to $K_0(\mathcal{C}_0)$, the diagram of simplicial spaces

$$
\begin{array}{ccc}
S_*\left(\mathcal{C}_0\right) & \longrightarrow & S_*\left(\mathfrak{C}\right) \\
\downarrow & & \downarrow \\
B_*K_0(\mathcal{C}_0) & \longrightarrow & B_*K_0(\mathfrak{C})
\end{array}
$$

is a homotopy pullback square. As in the previous lecture, we need to show that this remains true after geometric realization. Once again, this conclusion is not purely formal, because the spaces $B_*K_0(\mathfrak{C})$ are not connected (in fact, they are discrete). Our proof will proceed by taking advantage of some additional structure available in this situation: in this case, the coherently associative addition law on the spaces involved (given by the formation of coproducts in $\mathfrak{C}$).

**Notation 3.** Let $\mathcal{S}$ denote the $\infty$-category of spaces and let $\mathbf{Sp}$ denote the $\infty$-category of spectra. The formation of $0$th spaces determines a functor $\Omega^0 : \mathbf{Sp} \to \mathcal{S}$. The $\infty$-category $\mathbf{Sp}$ is stable, so that products and coproducts coincide. Consequently, every object $E \in \mathbf{Sp}$ can be regarded as a commutative monoid object of $\mathbf{Sp}$ in an essentially unique way. It follows that $\Omega^\infty$ determines a map $\mathbf{Sp} \to \mathbf{CAlg}(\mathcal{S})$, where $\mathbf{CAlg}(\mathcal{S})$ denotes the $\infty$-category of commutative monoid objects of $\mathcal{S}$: that is, the $\infty$-category of $E_\infty$-spaces.

It follows from abstract nonsense that the functor $\Omega^\infty : \mathbf{Sp} \to \mathbf{CAlg}(\mathcal{S})$ admits a left adjoint, which we will denote by $X \mapsto X^{gp}$. We will refer to $X^{gp}$ as the *group completion* of $X$. Tautologically, any $E_\infty$-space is equipped with an $E_\infty$-map $X \to \Omega^\infty X^{gp}$. Nontautologically, one can show that this map is a homotopy equivalence if and only if $X$ is grouplike: that is, $\pi_0X$ is a group.

Let $\mathfrak{C}$ be an $\infty$-category which admits finite coproducts. Then the formation of coproducts endows the underlying Kan complex $\mathfrak{C}^\Delta$ with the structure of an $E_\infty$-space. We will refer to the group completion $(\mathfrak{C}^\Delta)^{gp}$ as the *additive $K$-theory spectrum* of $\mathfrak{C}$ and denote it by $K_{\text{add}}(\mathfrak{C})$ (note that this conflicts with the notation of Lecture 14, where we used the same notation for the abelian group $\pi_0K_{\text{add}}(\mathfrak{C})$).

If $\mathfrak{C}$ is a pointed $\infty$-category which admits finite colimits, then each $\text{Gap}_{[n]}(\mathfrak{C})$ has the same property. It follows that each $S_n(\mathfrak{C})$ is an $E_\infty$-space which has a group completion $S_n(\mathfrak{C})^{gp}$. Since the geometric realization $|S_*\mathfrak{C}|$ is grouplike (it is connected, we have

$$
|S_*\mathfrak{C}| \simeq \Omega^\infty(|S_*\mathfrak{C}|^{gp})
$$

$$
\simeq \Omega^\infty(|S_*\mathfrak{C}|^{gp}).
$$

It follows that the diagram of Proposition 2 is obtained by applying $\Omega^\infty$ to a diagram of spectra

$$
\begin{array}{ccc}
|S_*\left(\mathcal{C}_0\right)^{gp}| & \longrightarrow & |S_*\left(\mathfrak{C}\right)^{gp}| \\
\downarrow & & \downarrow \\
|HB_*K_0(\mathcal{C}_0)| & \longrightarrow & |HB_*K_0(\mathfrak{C})|.
\end{array}
$$

The functor $\Omega^\infty$ preserves pullback squares, and the formation of geometric realizations of spectra commutes with pullbacks (since the $\infty$-category $\mathbf{Sp}$ is stable). It will therefore suffice to show that each of the diagrams

$$
\begin{array}{ccc}
S_n(\mathcal{C}_0)^{gp} & \longrightarrow & S_n(\mathfrak{C})^{gp} \\
\downarrow & & \downarrow \\
HB_nK_0(\mathcal{C}_0) & \longrightarrow & HB_nK_0(\mathfrak{C})
\end{array}
$$

is a pullback square. Replacing $\mathfrak{C}$ by $\text{Gap}_{[n]}(\mathfrak{C})$, we can reduce to the case $n = 1$. Proposition 2 is now reduced to the following “additive” version:
Proposition 4. Let \( \mathcal{C} \) be a stable \( \infty \)-category and let \( \mathcal{C}_0 \) be a full stable subcategory such that every object of \( \mathcal{C} \) is a direct summand of an object of \( \mathcal{C}_0 \). Then the diagram
\[
\begin{array}{ccc}
K_{\text{add}}(\mathcal{C}_0) & \longrightarrow & K_{\text{add}}(\mathcal{C}) \\
\downarrow & & \downarrow \\
HK_0(\mathcal{C}_0) & \longrightarrow & HK_0(\mathcal{C})
\end{array}
\]
is a homotopy pullback square.

Proof. This is a version of the group completion theorem. Let us indicate a proof. The spectra involved are connective, and the vertical maps are surjective on \( \pi_0 \). Consequently, it will suffice to show that the diagram of 0th spaces
\[
\begin{array}{ccc}
\Omega^\infty K_{\text{add}}(\mathcal{C}_0) & \longrightarrow & \Omega^\infty K_{\text{add}}(\mathcal{C}) \\
\downarrow & & \downarrow \\
K_0(\mathcal{C}_0) & \longrightarrow & K_0(\mathcal{C})
\end{array}
\]
is a pullback square. Let \( Z \) denote the inverse image of \( K_0(\mathcal{C}_0) \) in \( \Omega^\infty K_{\text{add}}(\mathcal{C}) \); we wish to show that the canonical map \( \theta : \Omega^\infty K_{\text{add}}(\mathcal{C}_0) \rightarrow Z \) is a homotopy equivalence. Since \( \Omega^\infty K_{\text{add}}(\mathcal{C}_0) \) and \( X \) are simple (they are infinite loop spaces), it will suffice to check that \( \theta \) induces an isomorphism in homology.

Consider the singular chain complexes
\[
A_0 = C_*(\Omega^\infty K_{\text{add}}(\mathcal{C}_0); \mathbb{Z}) \quad A = C_*(\Omega^\infty K_{\text{add}}(\mathcal{C}); \mathbb{Z}).
\]
Using the \( E_\infty \)-structures on the spaces involved, we can regard \( A_0 \) and \( A \) as \( E_\infty \)-algebras over \( \mathbb{Z} \). Similarly, we have \( E_\infty \)-algebras
\[
B_0 = C_*(\mathcal{C}_0^\Sigma; \mathbb{Z}) \quad B = C_*(\mathcal{C}^\Sigma; \mathbb{Z}).
\]
Note that \( B \) contains \( B_0 \) as a direct summand, and in fact we have a natural grading \( B = \bigoplus B_\alpha \) where \( \alpha \) ranges over the cosets of \( K_0(\mathcal{C}_0) \) in \( K_0(\mathcal{C}) \).

Using the universal property of the group completion, we see that \( A_0 \) can be obtained from \( B_0 \) by inverting all elements of the form \( [X] \in \mathbb{Z}[K_0(\mathcal{C}_0)] \simeq H_0(A_0) \) for \( X \in \mathcal{C}_0 \), and that \( A \) can be obtained from \( B \) by inverting all elements \( [X] \) for \( X \in \mathcal{C} \). However, since every object of \( \mathcal{C} \) is a direct summand of an object in \( \mathcal{C}_0 \), we only need to invert the classes \( [X] \) for \( X \in \mathcal{C}_0 \). We therefore have a canonical equivalence \( A \simeq A_0 \otimes_{B_0} B \). This equivalence determines a direct sum decomposition
\[
A \simeq \bigoplus_\alpha A_0 \otimes_{B_\alpha} B_\alpha,
\]
where the chain complex \( C_*(X; \mathbb{Z}) \) can be identified with the summand corresponding to \( \alpha = 0 \). From this description, it is clear that \( A_0 \simeq C_*(X; \mathbb{Z}) \).

Sometimes there is not much difference between \( K \)-theory and additive \( K \)-theory. Roughly speaking, we would expect this behavior in a situation where every cofiber sequence
\[
X' \rightarrow X \rightarrow X''
\]
_splits. However, this hypothesis is unreasonably strong in the context we have been discussing so far: for a cofiber sequence
\[
X \rightarrow * \rightarrow \Sigma(X)
\]
to split, we must have \( X \simeq * \). It will therefore be useful to consider a slightly more general setup:
Definition 5. An \(\infty\)-category with cofibrations is a pointed \(\infty\)-category \(\mathcal{C}\) with a distinguished class of morphisms, which we will call cofibrations, which satisfy the following axioms:

- All equivalences are cofibrations and the collection of cofibrations is closed under composition.
- For every object \(X\) in \(\mathcal{C}\), the canonical map \(* \to X\) is a cofibration.
- For a cofibration \(f : X \to X'\) and an arbitrary map \(X \to Y\), there exists a pushout square

\[
\begin{array}{ccc}
X & \rightarrow & X' \\
\downarrow & & \downarrow \\
Y & \rightarrow & Y'
\end{array}
\]

and the map \(g\) is also a cofibration.

Warning 6. We are using the term “cofibration” in order to follow the language of Waldhausen’s paper, but the notion of cofibration considered above does not \textit{a priori} have any relationship to the notion of cofibration in the language of model categories.

Example 7. Let \(\mathcal{C}\) be a pointed \(\infty\)-category. One way to try to satisfy the axiomatics of Definition 5 is to have as many cofibrations as possible. We can make \(\mathcal{C}\) into an \(\infty\)-category with cofibrations where \textit{all} morphisms are cofibrations if and only if \(\mathcal{C}\) has finite colimits.

Example 8. Let \(\mathcal{C}\) be a pointed \(\infty\)-category. Another way to try to satisfy the axiomatics of Definition 5 is to have as few cofibrations as possible. Note that if for any pair of objects \(X\) and \(Y\), the natural map \(* \to X\) is a cofibration and therefore there exists a pushout square

\[
\begin{array}{ccc}
* & \rightarrow & X \\
\downarrow & & \downarrow \\
Y & \rightarrow & X \vee Y
\end{array}
\]

where the lower horizontal map is a cofibration. Consequently, if \(\mathcal{C}\) is an \(\infty\)-category with cofibrations, then \(\mathcal{C}\) must have coproducts and every map of the form \(Y \to X \vee Y\) must be a cofibration.

Conversely, suppose that \(\mathcal{C}\) is a pointed \(\infty\)-category which admits finite coproducts. Then \(\mathcal{C}\) can be made into an \(\infty\)-category with cofibrations by declaring that a morphism \(f\) is a cofibration if and only if it is equivalent to a morphism of the form \(Y \to X \vee Y\); we will refer to such a morphism as a \textit{split cofibration}.

Let \(\mathcal{C}\) be an \(\infty\)-category with cofibrations. For each integer \(n\), we let \(\text{Gap}_{[n]}(\mathcal{C})\) denote the full subcategory of \(\text{Fun}(\mathcal{N}\{(i, j) \in [n] \times [n] : i \leq j\}, \mathcal{C})\) spanned by those functors \(X\) satisfying the following three conditions:

- For each \(i \leq j \leq k\), the natural map \(X(i, j) \to X(i, k)\) is a cofibration.
- For each \(i\), the object \(X(i, i)\) is zero.
- For each \(i \leq j \leq k\), the diagram

\[
\begin{array}{ccc}
X(i, j) & \rightarrow & X(i, k) \\
\downarrow & & \downarrow \\
* & \rightarrow & X(j, k)
\end{array}
\]

is a pushout square.
Arguing as in Lecture 16, we see that an object $X$ of $\text{Gap}_{[n]}(\mathcal{C})$ is determined by the diagram

$$X(0, 1) \to X(0, 2) \to \cdots \to X(0, n).$$

The only difference is that this time, we consider only those diagrams where each map is a cofibration.

**Definition 9.** Let $\mathcal{C}$ be an $\infty$-category with cofibrations. We let $S_\bullet(\mathcal{C})$ denote the simplicial space given by the formula $S_n(\mathcal{C}) = \text{Gap}_{[n]}(\mathcal{C})^\times$, where $\text{Gap}_{[n]}(\mathcal{C})$ is defined as above. We let $K(\mathcal{C})$ denote the space given by $\Omega|S_\bullet(\mathcal{C})|$. The simplicial space $S_\bullet(\mathcal{C})$ and $K(\mathcal{C})$ depend not only on the $\infty$-category $\mathcal{C}$, but also on the class of cofibrations chosen. For example, if $\mathcal{C}$ admits finite colimits and we declare that all morphisms are cofibrations (Example 7), then we recover the definitions of Lecture 16. If $\mathcal{C}$ admits finite coproducts and we use only the split cofibrations (Example 8), then we often recover additive $K$-theory $K_{\text{add}}(\mathcal{C})$.

**Theorem 10.** Let $\mathcal{C}$ be an $\infty$-category which admits finite products and finite coproducts, and assume that the homotopy category of $\mathcal{C}$ is additive (so that finite products and finite coproducts in $\mathcal{C}$ coincide). For example, we can take any stable $\infty$-category, or any subcategory of a stable $\infty$-category which is closed under direct sums. Regard $\mathcal{C}$ as an $\infty$-category with cofibrations as in Example 8 (allowing only split cofibrations). Then there is a canonical homotopy equivalence $K_{\text{add}}(\mathcal{C}) \to K(\mathcal{C})$ (where we abuse notation by identifying $K_{\text{add}}(\mathcal{C})$ with its 0th space).

We can identify the 0th space of $K_{\text{add}}(\mathcal{C})$ with the $\Omega|Y_\bullet|$, where $Y_\bullet$ is the simplicial space given by $Y_n = (\mathcal{C}^{\infty})^n$ (made into a simplicial space using the coproduct on $\mathcal{C}^{\infty}$). The map $K_{\text{add}}(\mathcal{C}) \to K(\mathcal{C})$ is then obtained from a map of simplicial spaces

$$Y_\bullet \to S_\bullet(\mathcal{C});$$

in degree $n$ this map is given by the construction

$$(C_1, \ldots, C_n) \mapsto (C_1 \to C_1 \oplus C_2 \to \cdots \to C_1 \oplus \cdots \oplus C_n).$$

We wish to show that the induced map of geometric realizations $|Y_\bullet| \to |S_\bullet(\mathcal{C})|$ is a homotopy equivalence. All of the spaces in sight admit $E_\infty$-structures coming from the formation of coproducts in $\mathcal{C}$. Arguing as before, we obtain homotopy equivalences

$$|Y_\bullet| \simeq \Omega^\infty |Y_\bullet|^{\text{gp}}$$

$$\simeq \Omega^\infty |Y_\bullet|^{\text{gp}}.$$  

$$|S_\bullet(\mathcal{C})| \simeq \Omega^\infty |S_\bullet(\mathcal{C})|^{\text{gp}}$$

$$\simeq \Omega^\infty |S_\bullet(\mathcal{C})|^{\text{gp}}.$$  

It will therefore suffice to show that for each $n \geq 0$, the map $Y_n \to S_n(\mathcal{C})$ induces a homotopy equivalence of spectra $Y_n^{\text{gp}} \to S_n(\mathcal{C})^{\text{gp}}$.

For simplicity, let us consider the case $n = 2$ (the general case is only notationally more difficult). The space $S_2(\mathcal{C})$ classifies morphisms $f : X \to X'$ which are split cofibrations in $\mathcal{C}$. Let $e : S_2(\mathcal{C}) \to \mathcal{C}^{\infty}$ be the map given by $e(X \to X') = X$, let $\pi : \mathcal{C}^{\infty} \times \mathcal{C}^{\infty} \to \mathcal{C}^{\infty}$ be projection onto the first factor, and let $i : \mathcal{C}^{\infty} \to \mathcal{C}^{\infty} \times \mathcal{C}^{\infty}$ be given by $X \mapsto (**, X)$. We then have a commutative diagram of $E_\infty$-spaces

$$\begin{array}{ccc}
\mathcal{C}^{\infty} & \xrightarrow{\pi} & \mathcal{C}^{\infty} \times \mathcal{C}^{\infty} \\
\downarrow{\iota} & & \downarrow{\pi}
\ast \xrightarrow{id} & \mathcal{C}^{\infty} & \xrightarrow{id} \mathcal{C}^{\infty}.
\end{array}$$

We wish to show that the upper left horizontal map becomes an equivalence after group completion. In other words, we wish to show that the square on the right becomes a pullback square after group completion. Since
the $\infty$-category of spectra is stable, this is equivalent to the assertion that the square on the right becomes a pushout square after group completion. The left square is clearly a pushout after group completion; it will therefore suffice to show that the outer square is a pushout after group completion. In fact, we claim that the left square is a pushout before group completion. In other words, we claim that $C \simeq \mathbb{C} \otimes \mathbb{C} \simeq \ast$ in the $\infty$-category of spaces, where the $\infty$-category $\mathcal{C} \simeq \mathbb{C}$ acts on $S_{2}(\mathcal{C})$ via the construction

$$a : \mathcal{C} \times S_{2}(\mathcal{C}) \to S_{2}(\mathcal{C}) \quad (C, X \to X') \mapsto (X \to X' \oplus C).$$

This is an assertion which can be tested fiberwise over $\mathcal{C} \simeq \mathbb{C}$. In other words, we are reduced to proving the following:

**Proposition 11.** In the situation of Theorem 10, fix an object $X \in \mathcal{C}$, and let $D$ denote the full subcategory of $\mathcal{C}_f$ spanned by the split cofibrations $X \to X'$. Let $C \simeq \mathbb{C}$ act on the space $D \simeq \mathbb{C}$ as above. Then the homotopy quotient

$$D \simeq \mathbb{C} \simeq C \simeq \mathcal{C} \simeq \ast$$

is contractible.

**Proof.** Note that the $\infty$-category $D$ admits finite coproducts (given by pushouts over $X$), so that $D \simeq \mathbb{C}$ is an $E_{\infty}$-space. We can regard the quotient $D \simeq \mathbb{C} / C \simeq \mathcal{C}$ as the cofiber of the natural map

$$f : \mathcal{C} \simeq \mathbb{C} \to D \simeq \mathbb{C} \simeq C \simeq (X \to X \oplus C)$$

in the $\infty$-category of $E_{\infty}$-spaces. By construction, the map $f$ is surjective on $\pi_0$ so that the quotient $D \simeq \mathbb{C} / C \simeq \mathcal{C}$ is connected. In particular, $D \simeq \mathbb{C} / C \simeq \mathcal{C}$ is grouplike, so it can be identified with (the 0th space of) its group completion. It will therefore suffice to show that the map $f$ induces an equivalence of group completions. Define $q : D \simeq \mathbb{C} \to \mathbb{C} \simeq C \simeq \mathcal{C}$ by the formula $q(X \to X') = X'/X$. The map $q$ is obviously a left homotopy inverse to $f$. To complete the proof, it will suffice to show that it is also a right homotopy inverse after group completion. In other words, we wish to show that the composite map

$$(f \circ q) : D \simeq \mathbb{C} \to D \simeq \mathbb{C}$$

$$(X \to X') \mapsto (X \to X \oplus (X'/X))$$

is homotopic to the identity map after group completion. In fact, we claim that $(f \circ q)$ is homotopic to the identity map id after adding a single copy of the identity map: that is, to any split cofibration $X \to X'$, we can functorially identify the split cofibrations

$$X \to X' \amalg_X X'$$

$$X \to X' \oplus (X'/X).$$

This identification follows from the additivity assumption on $\mathcal{C}$ (the “fold map” $X' \amalg_X X' \to X$ is split by the inclusion of either factor).

**References**