We begin by reviewing the definition of an $\infty$-category.

**Notation 1.** For every pair of integers $0 \leq i \leq n$, we let $\Lambda^n_i$ denote the simplicial subset of $\Delta^n$ given by the union of all those faces except the one opposite to the $i$th vertex. We will refer to a simplicial set of the form $\Lambda^n_i$ as a horn. We will say that it is an inner horn if $0 < i < n$, and otherwise an outer horn.

**Definition 2.** Let $X$ be a simplicial set. We will say that $X$ is an $\infty$-category if every map $f_0 : \Lambda^n_i \to X$ can be extended to an $n$-simplex $f : \Delta^n \to X$ provided that $0 < i < n$. (In other words, every inner horn in $X$ can be filled.)

**Remark 3.** Simplicial sets satisfying the requirements of Definition 2 are also referred to as quasi-categories or weak Kan complexes.

**Example 4.** Any Kan complex is an $\infty$-category (recall that a simplicial set $X$ is a Kan complex if any horn $f_0 : \Lambda^n_i \to X$ can be extended to an $n$-simplex of $X$).

**Example 5.** For any category $\mathcal{C}$, the nerve $N(\mathcal{C})$ is an $\infty$-category.

In fact, one has the following stronger assertion:

**Exercise 6.** Let $X$ be a simplicial set. Show that $X$ is isomorphic to the nerve of a category if and only if every inner horn $f_0 : \Lambda^n_i \to X$ can be extended uniquely to an $n$-simplex $f : \Delta^n \to X$.

In what follows, we will often abuse notation by identifying a category $\mathcal{C}$ with the $\infty$-category $N(\mathcal{C})$. This does not lose any information:

**Exercise 7.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Show that there is a bijective correspondence between the set of functors $F : \mathcal{C} \to \mathcal{D}$ and the set of maps of simplicial sets $N(\mathcal{C}) \to N(\mathcal{D})$. In other words, the formation of nerves induces a fully faithful embedding from the category of (small) categories to the category of simplicial sets.

The formation of nerves admits a left adjoint, which sends each simplicial set $X$ to a category which we will denote by $hX$. Concretely, the category $hX$ admits the following presentation by generators and relations:

- The objects of $hX$ are the vertices of $X$.
- For each edge $e$ of $X$ joining a vertex $x$ to a vertex $y$, there is a corresponding morphism $[e]$ from $x$ to $y$ in $hX$.
- If the edge $e$ is degenerate (so that $x = y$), then $[e] = \text{id}_x$. 
For every 2-simplex of $X$ given pictorially by the diagram

\[
\begin{array}{ccc}
  f & \downarrow g & \rightarrow h \\
  x & \downarrow & \rightarrow z,
\end{array}
\]

we have $[h] = [g] \circ [f]$ in $hx$.

We will refer to $hx$ as the homotopy category of $X$.

In the special case where $X$ is an $\infty$-category, the homotopy category $hx$ admits a more concrete description: all morphisms in $hx$ have the form $[e]$ for some edge $e$, and two edges $e$ and $e'$ (with the same initial and final vertices) satisfy $[e] = [e']$ if and only if they are homotopic (meaning that there exists a 2-simplex

\[
\begin{array}{ccc}
  e & \downarrow & \rightarrow e' \\
  x & \downarrow & \rightarrow y
\end{array}
\]

whose 0th face (joining $y$ to $y$) is degenerate).

Most of the basic concepts of category theory (commutative diagrams, limits and colimits, initial and final objects, functors, adjunctions) can be generalized to the setting of $\infty$-categories. We will henceforth make use of those generalizations, and refer the reader to [1] for more details.

In what follows, we will use the notation $C$ to denote an $\infty$-category (emphasizing the idea that $C$ is some sort of generalized category rather than a simplicial set). We refer to the vertices of $C$ as its objects and to the edges of $C$ as its morphisms.

**Definition 8.** Let $C$ be an $\infty$-category. A zero object of $C$ is an object $\ast$ which is both initial and final. We will say that $C$ is pointed if it has a zero object. If $C$ is pointed, then for every pair of objects $X$ and $Y$ there is a canonical morphism from $X$ to $Y$ given by the composition $X \rightarrow \ast \rightarrow Y$, which we refer to as the zero morphism.

**Notation 9.** Let $C$ be a pointed $\infty$-category with zero object $\ast$. Suppose that $C$ admits pushouts. For every morphism $f : X \rightarrow Y$ in $C$, we let $\text{cofib}(f)$ denote the pushout $Y \amalg_X \ast$. We refer to $f$ as the cofiber of $f$. In the special case where $Y = \ast$, we refer to $\text{cofib}(f)$ as the suspension of $X$ and denote it by $\Sigma X$. Note that we have a diagram

\[
\begin{array}{ccc}
  X & \rightarrow & \text{cofib}(f) \\
  \downarrow f & \rightarrow & \rightarrow \\
  Y & \rightarrow &
\end{array}
\]

where the composition is zero; we refer to such diagrams as cofiber sequences.

**Definition 10.** Let $C$ be a pointed $\infty$-category which admits pushouts. We let $K_0(C)$ denote the free abelian group on generators $[X]$, where $X$ is an object of $C$, modulo the relations given by $[X'] + [X''] = [X]$ whenever there is a cofiber sequence

\[
X' \rightarrow X \rightarrow X''
\]

in $C$.

**Remark 11.** Using the cofiber sequence

\[
\ast \rightarrow \ast \rightarrow \ast
\]

we deduce that $[\ast] = 0 \in K_0(C)$. Using the cofiber sequence

\[
X \rightarrow \ast \rightarrow \Sigma(X)
\]

we conclude that $[\Sigma(X)] = -[X]$ in $K_0(C)$. 

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Warning 12. Definition 10 is not interesting for “large” ∞-categories. For example, if C admits infinite coproducts, then any object X fits into a cofiber sequence

$$\coprod_{n \geq 1} X \to \coprod_{n \geq 0} X \to X$$

where the first two terms are equivalent to one another, so that \([X] = 0 \in K_0(C)\); since X was arbitrary, we have \(K_0(C) \cong 0\).

Example 13. Let C be the ∞-category of finite pointed spaces. Then \(K_0(C)\) is isomorphic to \(\mathbb{Z}\), the isomorphism being given by the “reduced” Euler characteristic

\([X] \mapsto \chi_{hyp}(X) = \chi(X) - 1\).

Example 14. Let \(R\) be a ring. A perfect complex over \(R\) is a bounded chain complex

$$\cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

where each \(P_i\) is a finitely generated projective \(R\)-module. The collection of perfect chain complexes over \(R\) can be organized into an ∞-category \(\text{Mod}^{perf}_R\). There is a natural map \(K_0(R) \to K_0(\text{Mod}^{perf}_R)\) which carries each finitely generated projective \(R\)-module \(P\) to the chain complex consisting of \(P\) in degree zero. One can show that this map is an isomorphism: it has an inverse which carries a chain complex \([P_*]\) to the alternating sum \(\sum_n (-1)^n [P_n]\).

Remark 15. Let C and D be pointed ∞-categories which admit pushouts and suppose we are given a functor \(F: C \to D\) which preserves finite colimits. Then \(F\) induces a group homomorphism \(K_0(C) \to K_0(D)\), given by \([X] \mapsto [F(X)]\).

Example 16. Let \(C\) be a pointed ∞-category which admits pushouts. Then the suspension functor \(\Sigma : C \to C\) satisfies the hypotheses of Remark 15, and induces the map \(K_0(C) \to K_0(C)\) given by multiplication by \(-1\).

Definition 17. We say that an ∞-category \(C\) is stable if it is pointed, admits pushouts, and the suspension functor \(\Sigma : C \to C\) is an equivalence of ∞-categories.

Remark 18. Let \(C\) be a pointed ∞-category which admits pushouts. We define the Spanier-Whitehead category \(SW(C)\) to be the direct limit

\(C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} C \to \cdots\).

Then \(SW(C)\) is stable, and is universal among stable ∞-category which receive a functor from \(C\) which preserves finite colimits. Moreover, \(K_0(SW(C))\) can be identified with the direct limit of the sequence

\(K_0(C) \xrightarrow{-1} K_0(C) \xrightarrow{-1} K_0(C) \to \cdots\),

and is therefore isomorphic to \(K_0(C)\).

In other words, for studying \(K_0\), there is no real loss of generality in assuming that we are working with stable ∞-categories.

In the next lecture, we will need the following result:

Proposition 19. Let \(C\) be a stable ∞-category and let \(C_0 \subseteq C\) be a full (stable) subcategory. Assume that every object of \(C\) is a direct summand of an object that belongs to \(C_0\). Then:

(a) The canonical map \(\alpha : K_0(C_0) \to K_0(C)\) is injective.

(b) An object \(C \in C\) belongs to \(C_0\) if and only if \([C]\) belongs to the image of \(\alpha\).

To prove Proposition 19, it will be convenient to introduce a variant of Definition 10.
Definition 20. Let $\mathcal{C}$ be a stable $\infty$-category. We let $K_{\text{add}}(\mathcal{C})$ denote the free abelian group generated by symbols $[X]$ where $X \in \mathcal{C}$, modulo the relations

$$[X] = [X'] + [X'']$$

if $X$ is equivalent to a direct sum $X' \oplus X''$.

Remark 21. In the situation of Definition 20, it is easy to see that we have $[X] = [Y]$ in $K_{\text{add}}(\mathcal{C})$ if and only if $X$ and $Y$ are stably equivalent: that is, if and only if there exists an object $Z \in \mathcal{C}$ such that $X \oplus Z$ is equivalent to $Y \oplus Z$.

We have an evident surjective map $K_{\text{add}}(\mathcal{C}) \to K_0(\mathcal{C})$; let us denote the kernel of this map by $I(\mathcal{C})$.

Lemma 22. In the situation of Proposition 19, the canonical map $K_0(\mathcal{C}) \to I(\mathcal{C})$ is surjective.

Proof. Note that $I(\mathcal{C})$ is generated by expressions of the form $\eta = [X] - [X'] - [X'']$, where $X' \to X \to X''$ is a cofiber sequence in $\mathcal{C}$. For any such cofiber sequence, we can choose objects $Y', Y'' \in \mathcal{C}$ such that $X' \oplus Y'$ and $X'' \oplus Y''$ belong to $\mathcal{C}_0$. We then have a cofiber sequence

$$X' \oplus Y' \to X \oplus Y' \oplus Y'' \to X'' \oplus Y''$$

where the outer terms belong to $\mathcal{C}_0$, so that the middle term does as well. It follows that $\eta = [X \oplus Y' \oplus Y''] - [X' \oplus Y'] - [X'' \oplus Y'']$ belongs to the image of the map $I(\mathcal{C}_0) \to I(\mathcal{C})$.

Proof of Proposition 19. It follows immediately from Remark 21 that the map $K_{\text{add}}(\mathcal{C}_0) \to K_{\text{add}}(\mathcal{C})$ is injective. Assertion (a) now follows by applying the snake lemma to the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & I(\mathcal{C}_0) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & I(\mathcal{C})
\end{array}
\quad
\begin{array}{ccc}
K_{\text{add}}(\mathcal{C}_0) & \longrightarrow & K_0(\mathcal{C}_0) \\
\downarrow & & \downarrow \\
K_{\text{add}}(\mathcal{C}) & \longrightarrow & K_0(\mathcal{C})
\end{array}
\longrightarrow 0.
$$

To prove (b), we note that the snake lemma implies that the natural map

$$K_{\text{add}}(\mathcal{C}) / \text{Im}(K_{\text{add}}(\mathcal{C}_0)) \to K_0(\mathcal{C}) / \text{Im} K_0(\mathcal{C}_0)$$

is an isomorphism of abelian groups. Consequently, if $X \in \mathcal{C}$ has the property that $[X]$ belongs to the image of $K_0(\mathcal{C}_0)$, then $[X]$ belongs to the image of $K_{\text{add}}(\mathcal{C}_0)$. It follows that there exists objects $Y, Y' \in \mathcal{C}_0$ such that $X \oplus Y \simeq Y'$, so that $X$ is equivalent to the cofiber of a map $Y \to Y'$ and therefore belongs to $\mathcal{C}_0$ as desired.

References