Lecture 20X-Ultraproducts

March 31, 2018

In this lecture, we review the theory of ultrafilters and ultraproducts.

**Definition 1.** Let $I$ be a set. An ultrafilter on $I$ is a collection $\mathcal{U}$ of subsets of $I$ satisfying the following conditions:

(a) The set $\mathcal{U}$ is closed under finite intersections. That is, the set $I$ belongs to $\mathcal{U}$, and for every $J, J' \in \mathcal{U}$, the intersection $J \cap J'$ also belongs to $\mathcal{U}$.

(b) The set $\mathcal{U}$ is closed upwards: that is, if $J \subseteq J'$ and $J$ is contained in $\mathcal{U}$, then $J'$ is also contained in $\mathcal{U}$.

(c) For every subset $J \subseteq I$, exactly one of the sets $J$ and $I - J$ belongs to $\mathcal{U}$.

**Exercise 2.** In Definition 1, show that (b) can be deduced from (a) and (c).

**Remark 3.** Let $I$ be a set. Then the datum of an ultrafilter $\mathcal{U}$ on $I$ is equivalent to the datum of a finitely additive measure $\mu : \{\text{Subsets of } I\} \to \{0, 1\}$; the equivalence is implemented by taking $\mathcal{U} = \{J \subseteq I : \mu(J) = 1\}$.

**Example 4 (Principal Ultrafilters).** Let $I$ be a set containing an element $i$, and let $\mathcal{U}_i$ be the collection of all subsets of $I$ which contain $i$. Then $\mathcal{U}_i$ is an ultrafilter on $I$. We refer to $\mathcal{U}_i$ as the principal ultrafilter associated to $i$.

**Exercise 5.** Let $\mathcal{U}$ be a collection of subsets of a set $I$. We say that $\mathcal{U}$ is a filter on $I$ if it satisfies conditions (a) and (b) of Definition 1. Show that if $\mathcal{U}$ is a filter on $I$ such that $\emptyset \not\in \mathcal{U}$, then $\mathcal{U}$ can be enlarged to an ultrafilter on $I$.

**Construction 6 (Ultraproducts).** Let $\{M_i\}_{i \in I}$ be a collection of sets indexed by a set $I$, and let $\mathcal{U}$ be an ultrafilter on $I$. We let $\prod_{i \in I} M_i / \mathcal{U}$ denote the direct limit

$$\lim_{J \in \mathcal{U}, i \in J} \prod_{i \in J} M_i.$$  

We will refer to $\prod_{i \in I} M_i / \mathcal{U}$ as the ultraproduct of the sets $M_i$ with respect to the ultrafilter $\mathcal{U}$.

**Exercise 7.** In the situation of Construction 6, suppose that each of the sets $M_i$ is nonempty. Show that the ultraproduct $\prod_{i \in I} M_i / \mathcal{U}$ can be identified with the quotient of $\prod_{i \in I} M_i$ by an equivalence relation $\sim$, where $\{x_i\}_{i \in I} \sim \{y_i\}_{i \in I}$ if $\{i \in I : x_i = y_i\}$ belongs to the ultrafilter $\mathcal{U}$ (in this case, we say that the sequences $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ agree almost everywhere with respect to $\mathcal{U}$).

Beware that this is not necessarily true if some $M_j$ is empty. In this case, the product $\prod_{i \in I} M_i$ is also empty. However, the ultraproduct $\prod_{i \in I} M_i / \mathcal{U}$ will be nonempty if the set $\{i \in I : M_i \neq \emptyset\}$ belongs to the ultrafilter $\mathcal{U}$.

**Example 8.** In the situation of Construction 6, suppose that $\mathcal{U} = \mathcal{U}_j$ is the principal ultrafilter associated to an element $j \in I$. Then the ultraproduct $\prod_{i \in I} M_i / \mathcal{U}$ can be identified with $M_j$.  

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Ultraproducts appear in mathematical logic because they behave well with respect to the truth of first-order formulas.

**Theorem 9** (Los’s Ultraproduct Theorem, Pretopos Version). Let $\mathcal{C}$ be a pretopos, let $\{M_i\}_{i\in I}$ be a collection of models of $\mathcal{C}$ indexed by a set $I$, and let $\mathcal{U}$ be an ultrafilter on $I$. Then the construction
\[
(C \in \mathcal{C}) \mapsto (\prod_{i \in I} M_i(C))/\mathcal{U}
\]
is also a model of $\mathcal{C}$.

**Corollary 10** (Los’s Ultraproduct Theorem, Classical Version). Let $T$ be a first-order theory in a language $\{P_j\}_{j \in J}$. Let $\{M_i\}_{i \in I}$ be a collection of models of $T$, and assume for simplicity that each $M_i$ is nonempty. Suppose we are given an ultrafilter $\mathcal{U}$ on the set $I$, and set $M = (\prod_{i \in I} M_i)/\mathcal{U}$. Regard $M$ as a structure for the language $L$ by declaring
\[
(M \models P_j(\{\vec{c}_i\}_{i \in I})) \iff \{i \in I : M_i \models P_j(\vec{c}_i)\} \in \mathcal{U}.
\]
Then $M$ is also a model of $T$. Moreover, for any formula $\varphi(\vec{x})$ in the language $L$, we have
\[
(M \models \varphi(\{\vec{c}_i\}_{i \in I})) \iff \{i \in I : M_i \models \varphi(\vec{c}_i)\} \in \mathcal{U}.
\]

**Proof.** Apply Theorem 9 to the syntactic category $\text{Syn}(T)$. (Note that the desired conclusion can be restated as $M[\varphi] \simeq (\prod_{i \in I} M_i[\varphi])/\mathcal{U}$.)

It is not difficult to give a direct proof of Theorem 9 (or Corollary 10): the essential point is that the formation of ultraproducts commutes with the formation of finite limits, finite coproducts, and images. However, we will give a different explanation of Theorem 9, which connects up with the material of the last few lectures.

For the remainder of this lecture, let $\mathcal{C}$ be a small pretopos. Recall that the category $\text{Pro}(\mathcal{C})$ has small limits and colimits.

**Proposition 11.**

1. The subcategory $\text{Pro}^{\text{wp}}(\mathcal{C}) \subseteq \text{Pro}(\mathcal{C})$ of weakly projective pro-objects of $\mathcal{C}$ has (possibly infinite) coproducts, which are preserved by the inclusion $\text{Pro}^{\text{wp}}(\mathcal{C}) \hookrightarrow \text{Pro}(\mathcal{C})$.

2. For every object $C \in \mathcal{C}$, the construction $M \mapsto M(C)$ determines a functor $\text{Pro}^{\text{wp}}(\mathcal{C})^{\text{op}} \to \text{Set}$ which preserves (possibly infinite) products: that is, it carries coproducts in $\text{Pro}^{\text{wp}}(\mathcal{C})$ to products of sets.

3. The category $\text{Stone}_\mathcal{C}$ has (possibly infinite) coproducts. Moreover, for each object $C \in \mathcal{C}$, the functor $(X, \mathcal{O}_X) \mapsto \mathcal{O}_X^C(X)$ carries coproducts in $\text{Stone}_\mathcal{C}$ to products of sets.

**Proof.** Recall that $\text{Pro}(\mathcal{C})$ can be defined as the opposite of the category $\text{Fun}_{\text{lex}}(\mathcal{C}, \text{Set})$ of left exact functors from $\mathcal{C}$ to $\text{Set}$. Since the class of left exact functors is closed under inverse limits, it follows that colimits in $\text{Pro}(\mathcal{C})$ are computed pointwise. In particular, given a collection of pro-objects $\{M_i\}_{i \in I}$, the coproduct $M = \bigoplus_{i \in I} M_i$ in the category $\text{Pro}(\mathcal{C})$ is given by the formula $M(C) = \prod_{i \in I} M_i(C)$. From this description, it is clear that that if each $M_i$ is weakly projective, then so is $M$ (note that a product of surjections in the category of sets is again a surjection). This proves (1) and (2), and assertion (3) is just a restatement.

**Example 12** (Ultrafilters). Let $\mathcal{C} = \text{Set}_{\text{fin}}$ be the category of finite sets, so that $\text{Stone}_{\mathcal{C}} \simeq \text{Stone}$ is the category of Stone spaces. Proposition 11 implies that the category Stone admits coproducts. Beware that the inclusion $\text{Stone} \hookrightarrow \text{Top}$ does not preserve coproducts: a coproduct of Stone spaces is Hausdorff and totally disconnected, but usually not compact.

For example, let $I$ be a set, and consider the coproduct $\bigoplus_{i \in I} \{i\}$, formed in the category $\text{Stone}$. We denote this coproduct by $\beta I$ and refer to it as the Stone–Čech compactification of $I$. It is characterized by the following universal property: there is a map $\rho : I \to \beta I$ such that composition with $\rho$ induces a bijection
\[
\text{Hom}_{\text{Top}}(\beta I, X) \to \prod_{i \in I} X
\]
for any Stone space $X$ (or, more generally, any compact Hausdorff space $X$). In particular, taking $X$ to be a two-point space, we obtain a bijection

$$\{\text{Clopen subsets of } \beta I\} \simeq \{\text{Arbitrary subsets of } I\}.$$ 

In other words, we can describe $\beta I$ as the spectrum of the Boolean algebra $P(I)$ of subsets of $I$. It follows that $\beta I$ can be identified with the set of Boolean algebra homomorphisms $\mu : P(I) \to \{0, 1\}$: that is, with the collection of all ultrafilters on $I$ (see Remark 3). The topology on $\beta I$ is generated by open (and closed) sets of the form

$$U_J := \{\mathcal{U} \in \beta I : J \in \mathcal{U}\},$$

where $J$ ranges over all subsets of $I$ (in fact, the construction $J \mapsto U_J$ implements the isomorphism of $P(I)$ with the Boolean algebra of clopen subsets of $\beta I$).

**Remark 13.** In the situation of Example 12, the canonical map $\rho : I \to \beta I$ carries each element $i \in I$ to the principal ultrafilter $\mathcal{U}_i$ of Example 4.

**Example 14** (Ultraproducts). Let us now return to the situation where $\mathcal{C}$ is an arbitrary small pretopos. Suppose we are given a collection of models $\{M_i \in \text{Mod}(\mathcal{C})\}_{i \in I}$. We can then regard each pair $\{(i), M_i\}$ as an object of $\text{Stone}_\mathcal{C}$, and form the coproduct

$$(X, \mathcal{O}_X) = \amalg_{i \in I}(\{i\}, M_i)$$

in $\text{Stone}_\mathcal{C}$.

Note that the forgetful functor $\text{Stone}_\mathcal{C} \to \text{Stone}$ preserves coproducts: it is given by the composition

$$\text{Stone}_\mathcal{C} \simeq \text{Pro}^{\text{wp}}(\mathcal{C}) \hookrightarrow \text{Pro}(\mathcal{C}) = \text{Fun}^{\text{lex}}(\mathcal{C}, \text{Set})^{\text{op}} \to \text{Fun}^{\text{lex}}(\text{Set}_{\text{fin}}, \mathcal{C})^{\text{op}} = \text{Pro}(\text{Set}_{\text{fin}}) = \text{Stone}$$

induced by the morphism of pretopoi $\text{Set}_{\text{fin}} \to \mathcal{C}$. It follows that we can identify the Stone space $X$ with the Stone-Čech compactification $\beta I$. In particular, the construction

$$(J \subseteq I) \mapsto U_J = \{\mathcal{U} \in \beta I : J \in \mathcal{U}\}$$

induces a bijection from the collection $P(I)$ of subsets of $I$ to the collection of clopen subsets of $X$. Unwinding the definitions, we see that $\mathcal{O}_X$ is given by the formula

$$\mathcal{O}^C_X(U_J) = \prod_{i \in J} M_i(C).$$

In particular, given a point $x \in X$ corresponding to an ultrafilter $\mathcal{U}$ on the set $I$, we have

$$\mathcal{O}^C_{X,x} = \lim_{x \in U_J} \mathcal{O}^C_X(U_J) = \lim_{J \in \mathcal{U}} \prod_{i \in J} M_i(C) = \left(\prod_{i \in I} M_i(C)\right)/\mathcal{U}.$$ 

**Proof of Theorem 9.** Let $\mathcal{C}$ be a pretopos, let $\{M_i\}_{i \in I}$ be a collection of models of $\mathcal{C}$ indexed by a set $I$, and let $\mathcal{U}$ be an ultrafilter on $I$. Forming the coproduct $(X, \mathcal{O}_X) = \amalg_{i \in I}(\{i\}, M_i)$ in $\text{Stone}_\mathcal{C}$, we observe that $\mathcal{U}$ can be identified with a point $x \in X \simeq \beta I$, and that the stalk $\mathcal{O}_{X,x}$ is a model of $\mathcal{C}$ given by the formula $C \mapsto (\prod_{i \in I} M_i(C))/\mathcal{U}$. □

We can summarize the situation informally as follows: given a collection of models $\{M_i\}_{i \in I}$ of a pretopos $\mathcal{C}$, we can construct a larger family of models parametrized by the Stone-Čech compactification $\beta I$, which assigns to each ultrafilter $\mathcal{U} \in \beta I$ the corresponding ultraproduct $(\prod_{i \in I} M_i)/\mathcal{U}$. 3
**Definition 15.** We will say that an object $M \in \text{Pro}(\mathcal{C})$ is *free* if it can be written as a coproduct $\coprod_{i \in I} M_i$ in $\text{Pro}(\mathcal{C})$, where each $M_i$ is a model of $\mathcal{C}$. Note that in this case, $M$ is automatically weakly projective.

We say that an object $(X, \mathcal{O}_X) \in \text{Stone}_\mathcal{C}$ is *free* if it corresponds to a free object of $\text{Pro}(\mathcal{C})$ under the equivalence $\text{Stone}_\mathcal{C} \simeq \text{Pro}^{\text{wp}}(\mathcal{C})$: that is, if it can be written as a coproduct

$$\coprod_{i \in I} (\{i\}, M_i)$$

in the category $\text{Stone}_\mathcal{C}$.

**Proposition 16.**

1. For every object $Z \in \text{Pro}(\mathcal{C})$, there exists an effective epimorphism $M \to Z$, where $M$ is free.

2. For every object $(X, \mathcal{O}_X)$, there exists a covering $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ in $\text{Stone}_\mathcal{C}$, where $(Y, \mathcal{O}_Y)$ is free.

**Proof.** To prove (1) we may assume without loss of generality that $Z$ is weakly projective. In this case, (1) and (2) are equivalent. Let us therefore consider (2). Fix an object $(X, \mathcal{O}_X)$ in $\text{Stone}_\mathcal{C}$, and form the coproduct

$$(Y, \mathcal{O}_Y) = \coprod_{x \in X} (\{x\}, \mathcal{O}_{X,x}).$$

We claim that the tautological map $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a covering. Using the criterion of Lecture 18X, we are reduced to showing that for each point $x \in X$, we can choose a point $y \in Y$ lying over $x$ for which the induced map of models $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is an isomorphism. Identifying $Y$ with the set $\beta X$ of ultrafilters on $X$, it suffices to choose $y$ to correspond to the principal ultrafilter $\mathcal{U}_x$; in this case, the canonical map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is an isomorphism (Example 8). \qed