Let us begin this lecture by continuing the analysis of ideals in von Neumann algebras. Let $A$ be a von Neumann algebra and let $I \subseteq A \subseteq B(V)$ be an ultraweakly closed $*$-ideal. We saw in the last lecture that $I$ can be regarded as a von Neumann algebra acting on the Hilbert space $IV \subseteq V$. In particular, the identity element of $I$ corresponds, in $B(V)$, to the orthogonal projection $e$ onto the closed subspace $IV \subseteq V$. Since $I$ is a left ideal, we have $AI = I$, so that $AIV \subseteq IV$. It follows that the subspace $IV \subseteq V$ is $A$-invariant, so that the projection $e$ also belongs to the commutant $A'$. We have proven the following:

**Proposition 1.** Let $A$ be a von Neumann algebra and let $I \subseteq A$ be a $*$-ideal which is ultraweakly closed. Then there exists a central Hermitian element $e \in A$ such that $e^2 = e$ and $I = eA = Ae = eAe$. It follows that $A$ decomposes as a product $I \times J$, where $J = (1 - e)A = A(1 - e) = (1 - e)A(1 - e)$.

Now suppose that $A$ is a nonunital $C^*$-algebra. We can always enlarge $A$ to a unital $C^*$-algebra $\tilde{A}$ by adding a unit. We then have an exact sequence (of nonunital $*$-algebra homomorphisms)

$$0 \to A \to \tilde{A} \xrightarrow{\phi} C \to 0.$$  

This gives an exact sequence of Banach spaces

$$0 \to A^{\vee \vee} \to \tilde{A}^{\vee \vee} \to C \to 0.$$  

We may therefore identify $A^{\vee \vee}$ with the kernel of the von Neumann algebra homomorphism $\psi : E(\tilde{A}) \simeq \tilde{A}^{\vee \vee} \to C$ determined by $\phi$. The analysis of Example ?? implies that this homomorphism determines von Neumann algebra isomorphism $E(\tilde{A}) \simeq C \times I$, where $I = \ker(\psi)$. As a Banach space, $I$ is isomorphic to $A^{\vee \vee}$.

We can summarize the situation as follows. The category of representations of $A$ as a nonunital $C^*$-algebra is equivalent to the category of representations of $\tilde{A}$ as a unital $C^*$-algebra, which is in turn equivalent to the category of representations of the von Neumann algebra $E(\tilde{A}) \simeq A^{\vee \vee} \times C$. The splitting reflects the fact that every nonunital representation $V$ of $A$ admits a canonical decomposition $V \simeq \tilde{A}V \oplus V_0$, where $\tilde{A}V$ is a nondegenerate representation of $\tilde{A}$ and $V_0$ is a trivial representation of $\tilde{A}$ (that is, every element of $\tilde{A}$ annihilates $V_0$). We have proven:

**Theorem 2.** Let $A$ be a nonunital $C^*$-algebra. Then the double dual $A^{\vee \vee}$ admits the structure of a von Neumann algebra. Moreover, there is a (nonunital) $*$-algebra homomorphism $A \to A^{\vee \vee}$ which determines an equivalence from the category of von Neumann algebra representations of $A^{\vee \vee}$ to the category of nondegenerate representations of $A$.

**Example 3.** Let $V$ be a Hilbert space and let $K(V) \subseteq B(V)$ denote the space of compact operators on $V$. Then $K(V)$ is a $*$-ideal in $B(V)$ which is closed in the norm topology. It is therefore a nonunital $C^*$-algebra (which has a unit if and only if $V$ is finite dimensional). The trace pairing

$$B^{\text{nc}}(V) \times B(V) \to C \quad \text{and} \quad B^{\text{nc}}(V) \times B(V) \to C$$
We can therefore write \( I \) belongs to \( A \). Let \( u \) denote a partial isometry, then \( uu^* \) and \( u^*u \) are both projection operators on \( V \) (exercise).

Suppose that \( T \in B(V) \) is an operator belonging to the commutant \( A' \). If \( v \in (fV)^\perp \), then

\[
(fw, Tv) = (T^* fw, v) = (TT^* w, v) = 0
\]

for all \( w \in V \), so that \( Tv \in (fV)^\perp \). It follows that

\[
u(Tv) = 0 = Tu(v).
\]

We also have

\[
u Tf(v) = ufT(v) = gT(v) = Tgf(v).
\]

It follows that \( Tu = uT \). Since this is true for all \( T \in A' \), we conclude that \( u \in A'' = A \). We have proven:

**Proposition 4.** Let \( A \) be a von Neumann algebra containing elements \( f \) and \( g \) with \( f^*f = g^*g \). Then \( A \) contains a partial isometry \( u \) satisfying \( uf = g \).

**Corollary 5** (Polar Decomposition). Let \( A \) be a von Neumann algebra containing an element \( f \). Then \( f \) admits a decomposition \( f = u|f| \), where \( u \) is a partial isometry and \( |f| \) denotes the unique positive square root of \( f^*f \).

**Corollary 6.** Let \( A \subseteq B(V) \) be a von Neumann algebra and let \( I \) be an ultraweakly closed left ideal. Then \( I = Ae \) for some projection \( e \).

**Proof.** Consider the intersection \( A_0 = I \cap I^* \). If \( x, y \in A_0 \), then \( xy \in xI \subseteq I \) and \( xy \in I^*y \subseteq I^* \), so that \( xy \in A_0 \). It follows that \( A_0 \) is a nonunital \(*\)-subalgebra of \( B(V) \), which is closed in the ultraweak topology. We can therefore write \( V \) as an orthogonal direct sum \( V_0 \oplus V_1 \), where \( A_0V_0 = 0 \) and \( V_1 \) is a nondegenerate representation of \( A_0 \). Let \( e \in B(V) \) denote the operator given by orthogonal projection onto \( V_1 \); it follows from the nonunital version of von Neumann’s theorem that \( e \in A_0 \). In particular, we deduce that \( e \in I \) so that \( Ae \subseteq I \). We claim that equality holds. To prove this, consider an arbitrary element \( x \in I \). Then \( x^*x \) belongs to \( I \cap I^* = A_0 \). Since \( A_0 \) is a \( C^* \)-algebra (in fact a von Neumann algebra) in its own right, \( x^*x \) has a unique positive square root in \( A_0 \), which we will denote by \( |x| \). Then \( |x| = |x|e \). The polar decomposition of \( x \) gives \( x = u|x| \) for some partial isometry \( u \in A \). Then \( x = u|x| = u|x|e \in Ae \), as desired.
The projection $e$ appearing in Corollary 6 is uniquely determined by the ideal $I$. To see this, let us use the notation $e_W$ to denote orthogonal projection onto a closed subspace $W \subseteq V$. If $A$ contains $e_W$, then $Ae_W$ is an ultraweakly closed left ideal in $A$ (it is the kernel of the map given by right multiplication by $1 - e_W = e_W\perp$). Note that a projection $e_W\perp$ belongs to $Ae_W$ if and only if $e_W\perp = e_W e_W\perp$; that is, if and only if $W' \subseteq W$. We may therefore characterize $e_W$ as the “largest” projection which belongs to the ideal $Ae_W$. We have proven the following.

**Corollary 7.** Let $A \subseteq B(V)$ be a von Neumann algebra. There is an order-preserving, one-to-one correspondence between ultraweakly closed left ideals of $A$ and closed subspaces $W \subseteq V$ such that $e_W \in A$. The correspondence is given by $W \mapsto Ae_W$.

**Corollary 8.** Let $A \subseteq B(V)$ be a von Neumann algebra and let $I = Ae_W$ be an ultraweakly closed left ideal in $A$. The following conditions are equivalent:

1. The ideal $I$ is a right ideal.
2. The projection $e_W$ belongs to the center of $A$.
3. The ideal $I$ is a $*$-ideal.

**Proof.** We have already seen that (3) $\Rightarrow$ (2). If $e_W$ is central then $Ae_W = e_W A$ is a right ideal, so that (2) $\Rightarrow$ (1). If $I$ is a right ideal, then $I^* = e_W A \subseteq I$, so that $I$ is a $*$-ideal; this proves (1) $\Rightarrow$ (3).

**Theorem 9.** Let $A$ be a $C^*$-algebra. Suppose there exists a Banach space $M$ and a Banach space isomorphism $A \cong M^\vee$. Assume further:

1. For each $a \in A$, left and right multiplication by $a$ are continuous with respect to the weak $*$-topology.

Then $A$ is isomorphic to a von Neumann algebra.

**Remark 10.** In the statement of Theorem 9, assumption (*) is actually unnecessary: just knowing that $A$ admits a Banach space predual guarantees that $A$ is isomorphic to a von Neumann algebra.

**Proof.** For every continuous linear map $\phi : A \rightarrow M^\vee$, there is an adjoint map $\phi' : M \rightarrow A^\vee$, which dualizes to give a map of Banach spaces $\hat{\phi} : A^{\vee\vee} \rightarrow M^\vee$. The map $\hat{\phi}$ is continuous with respect to the weak $*$-topologies on $A^{\vee\vee}$ and $M^\vee$, respectively, and fits into a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & M^\vee \\
\downarrow & & \downarrow \\
A^{\vee\vee} & \xrightarrow{\hat{\phi}} & M^\vee \\
\end{array}
$$

Moreover, $\hat{\phi}$ is uniquely determined by these properties (since $A$ is dense in $A^{\vee\vee}$ with respect to the weak $*$-topology).

Let us identify $A$ with $M^\vee$ and take $\phi$ to be the identity map. We then obtain a map of Banach spaces $r : A^{\vee\vee} \rightarrow A$, which is the identity on $A$. Let us denote the kernel of $r$ by $K \subseteq A^{\vee\vee}$. We have seen that $A^{\vee\vee}$ has the structure of a von Neumann algebra, and that the weak $*$-topology on $A^{\vee\vee}$ coincides with the ultraweak topology. It follows that $K \subseteq A^{\vee\vee}$ is ultraweakly closed.

Let $m_a : A \rightarrow A$ be the map given by left multiplication by an element $a \in A$, and let $\hat{m}_a : A^{\vee\vee} \rightarrow A^{\vee\vee}$ be given by left multiplication by the image of $A$. Consider the diagram

$$
\begin{array}{ccc}
A^{\vee\vee} & \xrightarrow{r} & A \\
\downarrow & \downarrow \hat{m}_a & \downarrow m_a \\
A^{\vee\vee} & \xrightarrow{r} & A \\
\end{array}
$$
Using assumption (*), we see that all of the maps appearing in this diagram are weak $\ast$-continuous. The diagram commutes when restricted to the image of $A$ in $A^{\vee\vee}$. Since this image is weak $\ast$-dense, we see that the diagram commutes. That is, $r$ commutes with left multiplication by $a \in A$. It follows that for $b \in K = \ker(r)$, we have $ab \in K$. The function $a \mapsto ab$ is ultraweakly continuous, and $K$ is ultraweakly closed. It follows that $ab \in K$ for all $a \in A^{\vee\vee}$: that is, $Ab \in K$. Since $b \in K$ was arbitrary, we conclude that $K$ is a left ideal in $A^{\vee\vee}$. The same argument proves that $K$ is a right ideal in $A^{\vee\vee}$, and therefore an ultraweakly closed $\ast$-ideal in $A^{\vee\vee}$ (Corollary 8). It follows that the quotient $A^{\vee\vee}/K$ inherits the structure of a von Neumann algebra (it is actually a direct factor of $A^{\vee\vee}$). We have a map of $\ast$-algebras

$$A \to A^{\vee\vee} \to A^{\vee\vee}/K$$

which is an isomorphism at the level of vector spaces, and therefore a $C^\ast$-algebra isomorphism. \qed