Let us recall the setting of the last lecture.

Notation 1. Let $A$ be a von Neumann algebra and $V$ a representation of $A$ containing a cyclic and separating vector $v$. We let $K$ denote the real Hilbert space given by the closure of the set $\{xv : x^* = x\}$, and $L$ the real Hilbert space given by the closure of the set $\{xv : x^* = -x\}$. Let $P$ and $Q$ denote the orthogonal projections onto $K$ and $L$, respectively (so that $P$ and $Q$ are real linear operators on $V$). Then we have a polar decomposition

$$P - Q = JP - QJ,$$

where $|P - Q| = (2 - P - Q)^{1/2}(P + Q)^{1/2}$. The operator $|P - Q|$ commutes with $J$, $P$ and $Q$, while $J$ satisfies

$$JP = (1 - Q)J \quad JQ = (1 - P)J.$$ 

It follows that $J(P + Q) = (2 - P - Q)J$, and therefore $J(P + Q)^{1/2}J = (2 - P - Q)^{1/2}.$

Note that in the present setting, $V$ is a complex Hilbert space and we have $L = iK$. It follows that $Qi = iP$. In particular, $i(P + Q) = iP + iQ = Qi + Pi = (P + Q)i$, so that $P + Q$ is a positive $C$-linear operator.

Construction 2. Let $T : V \to V$ be a normal $C$-linear bounded operator and let $\sigma(T)$ be its spectrum. Recall that for every bounded Borel measurable function $f$, we can define a new operator $f(T) \in B(V)$ using the Borel functional calculus. We will be particularly interested in the case where $T$ is a positive operator, so that $\sigma$ is a bounded subset of $\mathbb{R}_{\geq 0}$. For every complex number $z$ with $\Re(z) \geq 0$, we can take $f$ to be the function

$$t \mapsto t^z = \begin{cases} 0 & \text{if } t = 0 \\ e^{z \log(t)} & \text{if } t > 0. \end{cases}$$

(Note that this function is bounded on bounded subsets of $\mathbb{R}$.) In this case, we will denote the operator $f(T)$ by $T^z$. It belongs to every von Neumann subalgebra of $B(V)$ which contains $x$, and therefore commutes with every operator which commutes with $x$. Since the functional calculus is multiplicative, we have $T^{z+z'} = T^zT^{z'}$ for $z, z' \in \mathbb{C}$ with $\Re(z), \Re(z') \geq 0$. Note that $T^0$ is the projection operator which annihilates ker($T$); in particular, $T^0 = 1$ if $T$ is injective.

Since $f(T)^* = \overline{f(T^*)}$, we see that $(T^z)^* = T^{\overline{z}}$ when $T$ is a positive operator. In particular, $T^z$ is self-adjoint when $z$ is real. If $T$ is injective, then $T^{it}$ is inverse to $T^{-it} = (T^{it})^*$ for $t \in \mathbb{R}$, so that the operators $T^{it}$ are unitary.

If $0 \neq v \in V$ is an eigenvector for $T$ with eigenvalue $\lambda$ (that is, if $Tv = \lambda v$), then the positivity of $T$ implies that $\lambda \geq 0$. If $T$ is injective, we even have $\lambda > 0$. If $\Re(z) \geq 0$, we have $T^{z}(v) = \lambda^z v$.

Let us now return to the situation of interest: $V$ is a representation of $A$ with a cyclic and separating vector $v$. Now $P + Q$ and $2 - P - Q$ are injective positive self-adjoint operators. We can therefore define complex powers $(P + Q)^{z}$ and $(2 - P - Q)^{z}$ for $\Re(z) \geq 0$. Note that when $z = \frac{1}{2}$, this agrees with our
earlier definition (that is, we obtain the unique positive square roots of $P + Q$ and $2 - P - Q$). Since $J$ is an antiunitary map satisfying

$$J(P + Q)J^{-1} = 2 - P - Q,$$

we deduce that

$$J(P + Q)^zJ^{-1} = (2 - P - Q)^z$$

for every complex number $z$ satisfying $\Re(z) \geq 0$. We are primarily interested in the case where $z$ is purely imaginary. In this case, we obtain

$$J(P + Q)^{it}J^{-1} = (2 - P - Q)^{-it}$$
or $J(P + Q)^{it} = (2 - P - Q)^{-it}J$.

**Definition 3.** For every real number $t$, we let $\Delta^{it}$ denote the unitary operator given by $(2-P-Q)^{it}(P+Q)^{-it}$.

**Remark 4.** Recall that the unbounded operator $S : V \rightarrow V$ given by the closure of the operator $xv \mapsto x^*v$ has a polar decomposition $S = J\Delta^{1/2}$, where $\Delta^{1/2} = (2 - P - Q)^{1/2}(P + Q)^{-1/2}$. Thus Definition 3 is at least morally consistent with our earlier notation. In fact, one can extend Construction 2 to define complex powers of possible unbounded positive self-adjoint operators, so that the unitary operators $\Delta^{it}$ can be obtained directly from the unbounded operator $\Delta^{1/2}$.

**Remark 5.** Conjugation by $J$ carries $(P + Q)^t$ to $(2 - P - Q)^{-it}$. It follows that $J$ commutes with $\Delta^{it}$ for every real number $t$.

We can now state more fully the main results of Tomita-Takesaki theory.

**Theorem 6.** Let $A$ be a von Neumann algebra, let $V$ be a representation of $A$ containing a cyclic and separating vector $v$, and define $J$ and $\Delta^{it}$ as above. Then:

1. We have $A' = JA J$.
2. For every real number $t$, conjugation by $\Delta^{it}$ preserves the von Neumann algebras $A$ and $A'$.
3. If $c$ belongs to the center of $A$, then $JcJ = c^*$.

**Remark 7.** In the situation of Theorem 6, we get a one-parameter group of automorphisms $\{\sigma_t\}_{t \in \mathbb{R}}$ of $A$, given by $\sigma_t(x) = \Delta^{it}x\Delta^{-it}$. This family depends on the choice of $(V,v)$. Since the pair $(V,v)$ can be reconstructed from the state $\phi(x) = (xv,v)$, the construction $t \mapsto \sigma_t$ is called the modular flow associated to the state given by $\phi(x) = (xv,v)$.

The core of Theorem 6 is contained in the following result:

**Proposition 8.** In the situation of Theorem 6, for every real number $t$, conjugation by the operator $J\Delta^{it}$ carries $A'$ into $A$.

We will begin the proof of Proposition 8 in the next lecture. Let us assume Proposition 8 for the time being and explain how it leads to a proof of the whole of Theorem 6.

**Proof of Theorem 6.** Taking $t = 0$ in Proposition 8, we see that $JA'J \subseteq A$. To prove (1), it suffices to verify the reverse inclusion. Equivalently, we must show that $JAJ \subseteq A'$.

Note that the vector $v$ belongs to the real Hilbert space $K$, so that $Pv = v$. If $x \in A$ is skew-adjoint, then we have $(xv,v) = (v,x^*v) = (v,-xv) = -(xv,v)$. That is, the complex number $(xv,v)$ is purely imaginary. It follows that $v \in L^2$, so that $Qv = 0$. Thus $(P + Q)v = v$ and $(2 - P - Q)v = v$. It follows that $v$ is fixed by $(P + Q)^{1/2}$ and $(2 - P - Q)^{1/2}$, and therefore also by $|P - Q| = (P + Q)^{1/2}(2 - P - Q)^{1/2}$. Since $v$ is also fixed by $J|P - Q| = P - Q$, we deduce that $Jv = v$. 

2
Let $A_\mathbb{R}$ denote the set of self-adjoint elements of $A$. Since $J(K) = L^1$, the complex number $(Jxv, yv)$ is real for every pair of elements $x, y \in A_\mathbb{R}$. We therefore have

$$(Jxv, yv) = (yv, Jxv) = (xv, Jyv)$$

so that $(yJxv, v) = (v, xJyv)$. Both sides of this equation are $\mathbb{C}$ linear functions of $y$ and and $\mathbb{C}$-antilinear in $x$, so the identity holds for all $x, y \in A$.

Let $z' \in A'$. Using Proposition 8, we deduce that $Jz'J \in A$. Replacing $y$ by $yJz'J$ in the above identity, we get

$$(yJz'Jxv, v) = (v, xJyJz'Jv)$$

or $(yJz'Jxv, v) = (v, xJyJz'Jv)$. Since $z'$ commutes with $x$, the left hand side is given by

$$(yJz'xv, v) = (yJxz'v, v) = (Jxz'v, y^*v) = (Jy^*v, xz'v) = (x^*Jy^*v, z'v) = (x^*(Jy^*J)v, z'v).$$

Similarly, the right hand side is given by

$$(v, xJyJz'v) = (x^*v, JyJz'v) = (yJz'v, Jx^*v) = (Jz'v, y^*Jx^*v) = (Jy^*Jx^*v, z'v) = ((Jy^*J)x^*v, z'v).$$

Since $v$ is a separating vector, $A'v$ is dense in $V$. Hence the equalities

$$(x^*Jy^*Jv, z'v) = (\langle Jy^*Jv, Jy^*Jv \rangle)$$

imply that $x^*(Jy^*J)v = (Jy^*J)x^*v$ for all $x, y \in A$. Replacing $x^*$ by $xz$ and $y^*$ by $y$, we obtain

$$xz(JyJ)v = (JyJ)xzv.$$

The same identity gives $z(JyJ)v = (JyJ)zv$, so we can rewrite our equality as

$$x(JyJ)zv = (JyJ)xzv.$$

Since $v$ is a cyclic vector, $Av$ is dense in $V$. It follows that $x(JyJ) = (JyJ)x$ for all $x, y \in A$. This completes the proof of (1).

To prove (2), we note that for every real number $t$ we have

$$\Delta^itA\Delta^{-it} = \Delta^itJA'J\Delta^{-it} \subseteq A$$

by virtue of Proposition 8. The reverse inequality follows by replacing $t$ with $-t$.

We now prove (3). It is easy to see that the collection of those elements $c \in Z(A)$ which satisfy $JcJ = c^*$ is a $\mathbb{C}$-vector space which is closed in the norm topology. It will therefore suffice to show that $JcJ = c^*$ in the case where $c$ is a central projection of $A$. In this case, we can decompose $A$ as a product $A_- \times A_+$ and $V$ as a product $V_- \times V_+$, so that $c$ is given by orthogonal projection onto $V_-$. Unwinding the definitions, we note that $J$ decomposes into a pair of antiunitary involutions on $V_-$. In particular, $J$ commutes with the orthogonal projection onto $V_-$, so that $JcJ = c = c^*$.