Let $A$ be a $*$-algebra (in the purely algebraic sense). Then there exists a canonical $A$-$A$ bimodule $M$, given by $A$ itself. For $x \in A$, let $l_x, r_x : M \to M$ denote the operations given by left and right multiplication by $x$. The $*$-operator on $A$ determines a $\mathbb{C}$-antilinear map $J : M \to M$. Moreover, this map exchanges left and right multiplication in the following sense: for $x \in A$ and $y \in M$, we have

$$J(l_x y) = J(xy) = (xy)^* = y^* x^* = r_{x^*}(y^*) = r_{x^*} Jy.$$ 

That is, we have $r_{x^*} = J l_x J$.

This purely algebraic construction has the following features:

1. All left $A$-module maps from $M$ to itself are given by the right action of $A$ on $M$, and all right $A$-module maps from $M$ to itself and given by the left action of $A$ on $M$.

2. If $z$ belongs to the center of $A$, then $l_z$ and $r_z$ coincide.

Our goal in the next few lectures is to develop Tomita-Takesaki theory, which reconstructs an analogous picture in the setting of von Neumann algebras. If $A$ is a von Neumann algebra, we are generally interested in bimodules which are themselves Hilbert spaces. Usually, we cannot view $A$ as a bimodule over itself in this sense. However, we can construct something which is very analogous. Suppose that $V$ is a representation of $A$ containing a cyclic and separating vector $v$. In the last lecture, we studied the unbounded operator $S_0 : V \to V$ with domain $Av$, given by $S_0(xv) = x^* v$. We saw that this operator is closable, and that its closure $S$ is injective, densely defined, and has dense image. It follows that $S$ admits a spectral decomposition $S = J \Delta^\frac{1}{2}$, where $J$ is antiunitary and $\Delta^\frac{1}{2}$ is a self-adjoint unbounded $\mathbb{C}$-linear operator.

**Definition 1.** Let $F : V \to W$ be an unbounded operator between Hilbert spaces. Assume that $F$ is injective on the domain $V_0$ of $F$. Then we can define a new unbounded operator $F^{-1} : W \to V$ with domain $F(V_0)$, given by $F^{-1}(Fv) = v$. Note that the graphs of $F$ and $F^{-1}$ are identical (as subsets of $V \oplus W$, so that $F^{-1}$ is closed if and only if $F$ is closed. If not, then the closures of $F$ and $F^{-1}$ agree (provided both are defined). That is, if $F$ is closable and its closure $\overline{F}$ is injective, then $F^{-1}$ is closable and $\overline{F^{-1}} = \overline{F}^{-1}$.

In our situation, we have $S_0^{-1} = S_0$ (since the operation $x \mapsto x^*$ is its own inverse). It follows that $S^{-1} = S$. Let $\Delta^{-\frac{1}{2}}$ denote the inverse of $\Delta^\frac{1}{2}$. Then we get

$$J \Delta^{-\frac{1}{2}} = S^{-1} = \Delta^{-\frac{1}{2}} J^{-1} = J^{-1}(J \Delta^{-\frac{1}{2}} J^{-1})$$

From the uniqueness of the polar decomposition we deduce the following:

**Proposition 2.** The operator $J$ is an antiunitary involution (that is, $J^2 = \text{id}$), and $J \Delta^{-\frac{1}{2}} = \Delta^\frac{1}{2} J$.

Next week, we will prove the following result:

**Theorem 3.** Let $A$ be a von Neumann algebra, $V$ a representation of $A$ with a cyclic and separating vector $v$, and let $S = J \Delta^\frac{1}{2}$ be as before. Then:
(1) We have $A' = JAJ$. That is, conjugation by $J$ induces a conjugate-linear isomorphism of $A$ with its commutant $A'$.

(2) If $z \in Z(A)$, then $Jz = z^*J$.

**Remark 4.** In the situation of Theorem 3, we can define a right action of $A$ on $V$ by means of a map $\rho : A^{op} \to B(V)$ given by

$$\rho(x)(v) = Jx^*Jv.$$ 

Assertion (1) says that $\rho$ induces an isomorphism from $A^{op}$ to the commutant $A'$ of $A$, and assertion (2) says that this isomorphism is given by $z \mapsto z^*$ for $z \in Z(A) = A \cap A'$. In particular, the right action of $A$ on $V$ commutes with the left action of $A$ on $V$, so that we can regard $V$ as an $A$-$A$ bimodule.

We would like to say that Theorem 3 furnishes us with a canonical $A$-$A$ bimodule, analogous to the purely algebraic situation discussed above. However, the construction depends on a choice of pair $(V, v)$, where $V$ is a representation of $A$ and $v \in V$ is a cyclic and separating vector. Note that to give a pair $(V, v)$ where $v$ is cyclic is equivalent to giving the (ultraweakly continuous) state $\phi : A \to \mathbb{C}$, given by $\phi(x) = (xv, v)$. Note that $v$ is separating if and only if $(xv, xv) > 0$ for all nonzero $x \in A$. We can rewrite this as $\phi(x^*x) > 0$ for $x \neq 0$. That is, $v$ is separating if and only if the state $\phi$ is faithful, in the sense that it does not vanish on nonzero positive elements of $A$. We can therefore think of Theorem 3 as constructing an $A$-$A$ bimodule given a choice of faithful ultraweakly continuous state $\phi : A \to \mathbb{C}$.

Suppose we are given two different faithful (ultraweakly continuous) states $\phi, \psi : A \to \mathbb{C}$, from which we can construct a pair of representations $V_\phi$ and $V_\psi$ with cyclic vectors $v_\phi$ and $v_\psi$. We would like to compare these representations. To this end, consider the von Neumann algebra $B : M_2(A)$ of 2-by-2 matrices with coefficients in $A$. We define a linear functional $\phi \oplus \psi : B \to \mathbb{C}$ by the formula

$$(\phi \oplus \psi)(\begin{array}{cc} a & b \\ c & d \end{array}) = \phi(a) + \psi(d).$$

The associated inner product on $B$ is given by

$$(\begin{array}{cc} a & b \\ c & d \end{array}), (\begin{array}{cc} a' & b' \\ c' & d' \end{array}) \mapsto \phi(a'^*a) + \phi(c'^*c) + \psi(b'^*b) + \psi(d'^*d).$$

Let $W$ denote the Hilbert space completion of $B$ with respect to this inner product. We can think of the elements of $W$ as the space of matrices

$$(v \quad w)$$

with $v, v' \in V_\phi$ and $w, w' \in V_\psi$. From this description, we immediately see that $\phi \oplus \psi$ is a faithful state on $B$. We can think of the commutant $B'$ as consisting of matrices $(\begin{array}{cc} F & G \\ F' & G' \end{array})$ where $F \in A'_\phi$ belongs to the commutant of $A$ in $V_\phi$, $G' \in A'_\psi$, $F' \in \text{Hom}_A(V_\psi, V_\phi)$, and $G \in \text{Hom}_A(V_\psi, V_\psi)$. Let us apply Theorem 3 to the pairs $(A, V_\phi)$, $(A, V_\psi)$, and $(B, W)$. We obtain antiunitary involutions

$$J_\phi : V_\phi \to V_\phi \quad J_\psi : V_\psi \to V_\psi \quad J : W \to W.$$ 

Unwinding the definitions, we see that $J$ is given by the formula

$$J(\begin{array}{cc} v & w \\ v' & w' \end{array}) = (\begin{array}{cc} J_\phi v & U(v') \\ U^*(w) & J_\psi v \end{array}).$$
for some antiunitary isomorphisms \( U : V_\phi \to V_\psi \) and \( U' : V_\psi \to V_\phi \). Since \( J^2 = 1 \), we have \( U' = U^{-1} \). We now compute

\[
J \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} J \begin{pmatrix} v & w \\ v' & w' \end{pmatrix} = J \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} J_\phi v & U v' \\ U^{-1} w & J_\psi w' \end{pmatrix} = J \begin{pmatrix} 0 & 0 \\ J_\phi v & U v' \end{pmatrix} = \begin{pmatrix} 0 & U J_\phi v \\ 0 & J_\psi U v' \end{pmatrix}.
\]

Since the operator \( J \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} J \) belongs to \( B' \), it is given by some matrix \( \begin{pmatrix} F & G \\ F' & G' \end{pmatrix} \) above. It follows immediately that \( G = U J_\phi = J_\psi U \). From this, we deduce that \( U J_\phi \) is an \( A \)-linear unitary isomorphism \( \alpha \) from \( V_\phi \) to \( V_\psi \) satisfying \( J_\psi \circ \alpha = \alpha \circ J_\phi \). We have proven:

**Proposition 5.** Let \( A \) be a von Neumann algebra, let \( \phi \) and \( \psi \) be faithful ultraweakly continuous states on \( A \), let \( V_\phi \) and \( V_\psi \) be the associated representations, with antiunitary operators \( J_\phi \) and \( J_\psi \) given in Theorem 3. Then there exists an \( A \)-linear unitary isomorphism \( \alpha : V_\phi \to V_\psi \) such that \( \alpha \circ J_\phi = J_\psi \circ \alpha \). In particular, \( \alpha \) is an isomorphism of \( A-A \) bimodules.

**Remark 6.** The isomorphism \( \alpha \) is canonical. In fact, it is almost unique. Suppose we are given a pair of isomorphisms \( \alpha, \beta : V_\phi \to V_\psi \) satisfying the requirements of Proposition 5. Let \( \gamma = \beta^{-1} \circ \alpha \). Then \( \gamma \) is a unitary isomorphism of \( V_\phi \) with itself which commutes with the action of \( A \) and with \( J_\phi \). It therefore commutes with the action of the commutant \( A'_\phi = J_\phi A J_\phi \). It follows that \( \gamma \in A \cap A'_\phi = Z(A) \). Theorem 3 then gives \( J_\phi \circ \gamma = \gamma \circ J_\phi \), so that \( \gamma = \gamma^* \). Since \( \gamma \) is unitary, we deduce that \( \gamma^2 = 1 \). If \( A \) is a factor, this means that \( \gamma = \pm 1 \): that is, the isomorphisms \( \alpha \) and \( \beta \) differ by at most a sign.

**Definition 7.** Let \( A \) be a von Neumann algebra which admits an ultraweakly continuous faithful state \( \phi \) (for example, any separable von Neumann algebra). We let \( L^2(A) \) denote the Hilbert space \( V_\phi \), and \( J : L^2(A) \to L^2(A) \) the antiunitary isomorphism appearing in Theorem 3. It follows from the above considerations that the pair \((L^2(A), J)\) is independent of the choice of \( \phi \) up to isomorphism.

**Remark 8.** Since the isomorphism \( \alpha \) appearing in Proposition 5 is not quite unique, one might worry that \( L^2(A) \) is not quite well-defined. However, although \( \alpha \) is not unique it is nevertheless canonical. That is, given a triple of ultraweakly continuous faithful states \( \phi_0, \phi_1, \phi_2 : A \to \mathbb{C} \), the diagram of Hilbert spaces and unitary isomorphisms

\[
\begin{array}{ccc}
V_\phi & \xrightarrow{\alpha_{01}} & V_{\phi_1} \\
V_{\phi_0} & \xrightarrow{\alpha_{02}} & V_{\phi_2} \\
\end{array}
\]

is actually commutative. One can prove this by applying Theorem 3 to the state \( \phi_0 \oplus \phi_1 \oplus \phi_2 \) on \( M_3(A) \).