Let $A$ be a von Neumann algebra equipped with an ultraweakly continuous state $\phi : A \to \mathbb{C}$. We can use $\phi$ to construct a Hilbert space $V_\phi$, given by completing $A$ with respect to the inner product

$$(x, y) = \phi(y^* x).$$

Then $V_\phi$ is a representation of $A$, equipped with a cyclic vector which we will denote by $v$ (so that the map from $A$ into $V_\phi$ is given by $x \mapsto xv$).

**Question 1.** In terms of the above construction, what does it mean for the state $\phi$ to be tracial?

**Definition 2.** Let $V$ be a Hilbert space. We say that a map $J : V \to V$ is **antiunitary** if $J$ is $\mathbb{C}$-antilinear and satisfies

$$(v, w) = (Jv, Jw) = (Jw, Jv)$$

for each $v, w \in V$. We say that $J$ is an **antiunitary involution** if $J^2$ is the identity map. In this case, we can rewrite the above identity as

$$(Jv, w) = (Jw, v).$$

**Proposition 3.**

1. Let $\phi$ be an (ultraweakly continuous) finite trace on a von Neumann algebra $A$. Then the construction $xv \mapsto x^* v$ extends continuously to an antiunitary involution on the Hilbert space $V_\phi$.

2. Let $A$ be a von Neumann algebra and let $V \in \text{Rep}(A)$. Let $v \in V$ be a vector, and suppose that there exists an antiunitary map $J : V \to V$ such that $J(xv) = x^* v$ for all $x \in A$. Then the map $\phi : A \to \mathbb{C}$ given by $\phi(x) = (xv, v)$ is a finite trace on $A$.

**Proof.** Suppose first that $\phi$ is a finite trace, and let $v \in V_\phi$ be the canonical vector. We then have

$$(xv, yv) = \phi(y^* x) = \phi(xy^*) = (y^* v, x^* v) = (x^* v, y^* v),$$

so that the map $xv \mapsto x^* v$ extends to an antiunitary map from $V_\phi$ to itself. This map is obviously an involution.

Now suppose that $V$ is an arbitrary representation of $A$ equipped with a vector $v \in V$ and an antiunitary map $J$ satisfying the requirements of (2). We then have

$$\phi(xy) = (xyv, v) = (yv, x^* v) = (Jx^* v, Jyv) = (xv, y^* v) = (yxv, v) = \phi(yx)$$

so that $\phi$ is a trace.

**Remark 4.** In the situation of part (2) of Proposition 3, suppose in addition that $v \in V$ is a cyclic vector. Then $V$ is canonically isomorphic to the Hilbert space $V_\phi$ constructed from the state $\phi$. Since $J^2 xv = J(x^* v) = xv$ for all $x \in A$, we conclude that $J^2$ is the identity on $V$: that is, $J$ is automatically involutive.
Let φ be a finite ultraweakly continuous trace on A, and regard $V_\phi$ as a representation of A. For each $x \in A$, let $l_x$ denote the operator on $V_\phi$ induced by the left action of A on itself. That is, we have

$$l_x(yv) = (xy)v.$$ 

We then compute

$$(Jl_x)(yv) = Jl_x(y^*v) = J((xy^*)v) = (yx^*)v = r_x(yv),$$

where $r_x : V_\phi \to V_\phi$ is given by continuously extending the map given by right multiplication by $x^*$. If we regard $l$ as a map of $\ast$-algebras $A \to B(V_\phi)$, then then conjugation by $J$ carries $l$ to another map of $\ast$-algebras $A^{\text{op}} \to B(V_\phi)$. Let $A'$ denote the commutant of A in $B(V_\phi)$. Since the operators $l_x$ and $r_y$ commute for $x, y \in A$, we can regard $r$ as a map of von Neumann algebras $A^{\text{op}} \to A'$.

**Remark 5.** More generally, let A be any von Neumann algebra equipped with a representation $\rho : A \to B(V)$, a cyclic vector $v \in V$, and an antiunitary map $J : V \to V$ satisfying $J(xv) = x^*v$ for $x \in A$. Then $V \cong V_\phi$ where $\phi$ is the trace given by $\phi(x) = (xv, v)$. It follows that $x \mapsto J\rho(x).J$ determines a map $\rho' : A^{\text{op}} \to B(V)$, whose image is contained in the commutant $A'$ of the image of the original map $\rho$.

We now prove a result which was promised several lectures back:

**Proposition 6.** Let $\phi$ be an ultraweakly continuous finite trace on a von Neumann algebra $A$, and let $V_\phi$ denote the corresponding representation. Then the above construction induces a surjection $\rho : A^{\text{op}} \to A'$. In particular, if $\phi$ is faithful, we get an isomorphism $A^{\text{op}} \cong A'$.

**Proof.** Replacing A by a direct factor if necessary, we may assume without loss of generality that $\phi$ is faithful. We will identify A with its image in $B(V_\phi)$ (under the representation given by the left action of A on itself). Let us regard $V_\phi$ as a representation of $A'$. For each $x \in A$, the operator $r_x$ (given by $yv \mapsto yxv$) belongs to $A'$. In particular, $xv = r_x(v) \in A'v$, so that $Av \subseteq A'v$. It follows that v is a cyclic vector for $V_\phi$, regarded as a representation of $A'$.

Let $J : V_\phi \to V_\phi$ be the antiunitary involution constructed above (given by $xv \mapsto x^*v$ for $x \in A$). We claim that $J$ satisfies the hypothesis of part (2) of Proposition 3, with respect to the action of $A'$ on $V_\phi$. That is, we claim that if $F \in A'$, then we have $JF(v) = F^*v$. Since $V_\phi$ is topologically generated by elements of the form $xv$, it will suffice to show that

$$(JF(v), xv) = (F^*(v), xv)$$

for each $x \in A$. We now compute

$$(JF(v), xv) = (Jxv, F(v)) = (x^*v, F(v)) = (v,xF(v)) = (v,F(xv)) = (F^*(v), xv).$$

Applying Remark 5, we deduce that for each $F \in A'$, we have $JF \in A'' = A$. It follows that $F = JxJ$ for some $x \in A$: that is, $F$ belongs to the image of the map $A^{\text{op}} \to A'$.

**Remark 7.** Let $A$ be a von Neumann algebra acting on a Hilbert space $V$, let $v \in V$ be a cyclic vector, and let $J : V \to V$ be an antiunitary map. The equation $J(xv) = x^*v$ is equivalent to the assertion that, for each $w \in V$, we have

$$(Jxv, w) = (x^*v, w).$$

We can rewrite the left hand side as $(Jw, xv)$ and the right hand side as $(v, xw)$. Consequently, the hypothesis of (2) of Proposition 3 can be written as

$$(Jw, xv) = (v, xw).$$
for all \( x \in A \). Both sides of this equation are ultraweakly continuous functions of \( x \). Consequently, if \( A_0 \subseteq A \) is a \(*\)-algebra generating \( A \), then it suffices to verify the equation \( J(xv) = x^*v \) for elements \( x \in A_0 \).

**Definition 8.** Let \( G \) be a locally compact group, and choose a Haar measure \( \mu \) on \( G \). We let \( L^2(G) = L^2(G, \mu) \) denote the space of square-integrable functions on \( G \). Note that \( G \) acts on \( L^2(G) \) by left translation. The *group von Neumann algebra of \( G \)* is the von Neumann algebra in \( B(L^2(G)) \) generated by the image of \( G \). We will denote this von Neumann algebra by \( A(G) \).

In other words, the group von Neumann algebra \( A(G) \) is the ultraweak closure of the image of the group ring \( \mathbb{C}[G] \) inside \( B(L^2(G)) \) (we regard \( \mathbb{C}[G] \) as a \(*\)-algebra, with involution given by \( (\sum \lambda_g g)^* = \sum \lambda_g g^{-1} \)).

Let us now suppose that the group \( G \) is discrete. Let \( e \in G \) be the identity element. For each \( g \in G \), let \( \chi_g \in L^2(G) \) denote the characteristic function of the set \( \{ g \} \), so that
\[
\chi_g(h) = \begin{cases} 
1 & \text{if } g = h \\
0 & \text{if } g \neq h.
\end{cases}
\]
Let \( l_g, r_g : L^2(G) \to L^2(G) \) be the maps given by left and right translation, so that
\[
l_g(\chi_h) = \chi_{gh} \quad r_g(\chi_h) = \chi_{hg}.
\]
We let \( J : L^2(G) \to L^2(G) \) denote the antiunitary map given by \( (Jf)(g) = \overline{f(g^{-1})} \) (so that \( J(\chi_g) = \chi_{g^{-1}} \)).

Note that
\[
J(l_g \chi_e) = J(\chi_g) = \chi_{g^{-1}} = l_{g^{-1}} \chi_e = l_g^* \chi_e.
\]
Since the elements \( l_g \) generate \( A(G) \) as a von Neumann algebra, it follows from Remark 7 that the triple \((L^2(G), J, \chi_e)\) satisfies the hypothesis of part (2) of Proposition 3. In particular, we obtain a finite trace
\[
\phi : A(G) \to \mathbb{C},
\]
given by \( \phi(x) = (x\chi_e, \chi_e) \). We can characterize \( \phi \) as the unique ultraweakly continuous function whose restriction to the group algebra \( \mathbb{C}[G] \) is given by
\[
\phi(\sum \lambda_g g) = \lambda_e.
\]

**Proposition 9.** Let \( G \) be a nontrivial discrete group in which every non-identity conjugacy class is infinite. Then \( A(G) \) is a factor of type \( II_1 \).
Proof. Let \( x = \sum \lambda_g g \) be an element of \( A(G) \). If \( x \) is central, then it commutes with every element of \( G \). This implies that the function \( g \mapsto \lambda_g \) is invariant under conjugation. Since every nontrivial conjugacy class of \( G \) is infinite, the square summability of the coefficients \( \lambda_g \) implies that \( \lambda_g = 0 \) for \( g \neq e \). Thus the center of \( A(G) \) is one-dimensional, so that \( A(G) \) is a factor.

We have seen above that the von Neumann algebra \( A(G) \) admits a finite trace, and is therefore finite. Consequently, \( A(G) \) is either of type I or type II. If it is of type I, then it is isomorphic to \( B(V) \) for some Hilbert space \( V \). The finiteness of \( A(G) \) then implies that \( V \) is finite-dimensional, so that \( A(G) \simeq B(V) \) is finite dimensional. But \( G \) contains an infinite conjugacy class, so that the group algebra \( \mathbb{C}[G] \) is infinite dimensional. Since this group algebra injects into \( A(G) \), we obtain a contradiction. It follows that \( A(G) \) is a factor of type II. Since it is finite, it is of type \( II_1 \).

Corollary 10. There exist factors of type II.

Proof. It is easy to find examples of groups \( G \) satisfying the requirements of Proposition 9. For example, we can take \( G \) to be a free group on two generators.