Our goal in this lecture is to prove the following result, which was asserted without proof in the last lecture:

**Theorem 1** (Ryll-Nardzewski). Let $M$ be a Banach space, let $K$ be a convex subset of $M$ which is compact with respect to the weak topology on $M$, and let $G$ be a group of isometries of $M$ which preserves $K$. Then there is an element of $K$ which is fixed by the action of $G$.

As a warm-up, we prove the following simpler result:

**Proposition 2.** Let $M$ be a Banach space, let $K$ be a convex subset of $M$ which is compact with respect to the weak topology on $M$, and let $F : M \to M$ be a bounded linear map which preserves $K$ (not necessarily an isometry). Then there is an element $x \in K$ satisfying $F(x) = x$.

**Proof.** For each integer $n$, let $F_n$ denote the bounded linear map

$$x \mapsto \frac{1}{n}(x + F(x) + F^2(x) + \cdots + F^{n-1}(x)).$$

Since $K$ is convex, each of these maps carries $K$ into itself. Let $K_n = F_n(K) \subseteq K$, so that $K_n$ is a weakly compact subset of $K$. We first claim that the intersection $\bigcap K_n$ is nonempty. Since $K$ is weakly compact, it will suffice to show that each finite intersection $K_{n_1} \cap \cdots \cap K_{n_m}$ is nonempty. This follows from the observation that this intersection contains

$$F_{n_1}(F_{n_2}(\cdots (F_{n_m}(x))))$$

for each $x \in K$ (note that the operators $F_j$ commute with one another).

Let $x \in \bigcap K_n$. For each integer $y$, we can write $x = F_n(y)$ for some $y \in K$. It follows that

$$F(x) - x = F\left(\frac{y + \cdots + F^{n-1}y}{n}\right) - \frac{y + \cdots + F^{n-1}y}{n} = \frac{1}{n}(F^n(y) - y) \in \frac{1}{n}(K - K).$$

Since $K - K$ is weakly compact, every weakly open neighborhood of 0 in $M$ contains $\frac{1}{n}(K - K)$ for $n$ large enough. It follows that $F(x) - x$ belongs to every weakly open neighborhood of the origin, so that $F(x) = x$.

We now turn to the proof of Theorem 1. The first observation is:

(a) We may assume without loss of generality that $G$ is finitely generated.
Write $G$ as a union of finitely generated subgroups $G_{\alpha}$. Then the fixed point set $K^{G}$ is given by the intersection $\bigcap_{\alpha} K^{G_{\alpha}}$. By compactness, if each $K^{G_{\alpha}}$ is nonempty, then $K^{G}$ will be nonempty.

Let us suppose that $G$ contains elements $g_{1}, \ldots, g_{n} \in G$. Let $F : M \to M$ be the linear map given by $F(x) = \frac{g_{1}(x) + \cdots + g_{n}(x)}{n}$, so that $F$ carries $K$ into itself. Using Proposition 2, we can choose an element $x \in K$ such that $F(x) = x$. We will prove that $g_{i}(x) = x$ for all $i$. Taking the $g_{i}$ to be a set of generators for the group $G$, we will obtain a proof Theorem 1.

Suppose otherwise. We may assume without loss of generality that there exists some integer $1 \leq m \leq n$ such that $g_{i}(x) \neq x$ for $i \leq m$, and $g_{i}(x) = x$ for $i > m$. Then

$$x = F(x) = \frac{1}{n} \left( \sum_{1 \leq i \leq m} g_{i}(x) \right) + \frac{n - m}{n} x$$

so that

$$\frac{m}{n} x = \frac{1}{n} \sum_{1 \leq i \leq m} g_{i}(x)$$

and therefore $x$ is fixed by the operator $y \mapsto \frac{g_{1}(y) + \cdots + g_{m}(y)}{m}$. We may therefore replace the sequence \{g_{1}, \ldots, g_{m}\} by \{g_{1}, \ldots, g_{m}\}, and thereby reduce to the case where $g_{i}(x) \neq x$ for all $i$.

To obtain a contradiction, we are free to replace $G$ by the group generated by the elements $g_{1}, \ldots, g_{m}$, and $K$ by the closed convex hull of the orbit $Gx \subseteq K$ (in the weak topology). In particular, $K$ is contained in the closed subspace of $M$ generated by a countable set of vectors. Replacing $M$ by this closed subspace, we may assume that $M$ is separable.

Choose a real number $\varepsilon > 0$ such that $||g_{i}(x) - x|| > \varepsilon$ for each $i$. We will need the following technical lemma:

**Lemma 3.** There exists a weakly compact convex subset $K' \subseteq K$ such that the difference $K - K'$ has diameter $\leq \varepsilon$.

Let us assume Lemma 3 for the moment. Since $K$ is the closed convex hull of $Gx$ and $K' \subseteq K$ is closed and convex, there must exist an element $h \in G$ such that $hx \notin K'$. Then

$$hx = hF(x) = \frac{hg_{1}(x) + \cdots + hg_{n}(x)}{n} \notin K'.$$

Since $K'$ is convex, this implies that $hg_{i}(x) \notin K'$ for some $i$. Then $hx, hg_{i}(x) \in K - K'$. Since $K - K'$ has diameter $\leq \varepsilon$, we conclude that $||hg_{i}(x) - h(x)|| \leq \varepsilon$. Since $h$ is an isometry, we obtain $||g_{i}(x) - x|| \leq \varepsilon$, contradicting our assumption.

It remains to prove Lemma 3. Let $E$ denote the set of extreme points of $K$ (that is, points which do not lie on the interior of any line segment contained in $K$). Since $K$ is compact (in the weak topology), the Krein-Milman theorem asserts that $K$ is the closed convex hull of $E$. Let $E \subseteq K$ denote the weak closure of $E$. Let $B$ denote the closed ball of radius $\frac{\varepsilon}{3}$ around the origin. Note that $B$ is also closed in the weak topology (since $y \in B$ if and only if $|\phi(y)| \leq \frac{\varepsilon}{3}$ for all linear functionals $\phi$ of norm 1). Since $M$ is separable, there exists a countable collection of points $y_{i} \in M$ such that the sets $y_{i} + B$ cover $M$. In particular, the intersections

$$(y_{i} + B) \cap \overline{E}$$

give a countable covering of $\overline{E}$ by weakly closed subsets. Since $\overline{E}$ is weakly compact, the Baire category theorem implies that one of the sets $(y_{i} + B) \cap \overline{E}$ has nonempty interior $U$ in $\overline{E}$ (with respect to the weak topology).

Let $K_{1}$ be the closed convex hull of $\overline{E} - U$ and let $K_{2}$ be the closed convex hull of $(y_{i} + B) \cap \overline{E}$. Then $K_{1}$ and $K_{2}$ are closed convex subsets of $K$. Since $K$ is the closed convex hull of $E \subseteq (\overline{E} - U) \cup (y_{i} + B)$, it is the convex join of $K_{1}$ and $K_{2}$. That is, $K$ can be described as the image of the map

$$K_{1} \times K_{2} \times [0, 1] \to M$$
\[(v, w, t) \mapsto tv + (1 - t)w.\]

For \(\delta > 0\), let \(K(\delta)\) denote the image of the restriction of this map to \(K_1 \times K_2 \times [\delta, 1]\). We claim that if \(\delta\) is small enough, then \(K(\delta)\) has the desired properties. It is clear that each \(K_\delta\) is a weakly closed convex subset of \(K\). We are therefore reduced to proving two things:

(i) For \(\delta\) sufficiently small, the set \(K - K(\delta)\) has diameter \(\leq \epsilon\). Note that \(K\) is contained in a ball of some finite radius \(C\) (when regarded as a set of linear operators on \(M^\vee\), \(K\) is pointwise bounded by compactness, hence uniformly bounded). If \(y, y' \in K - K_\delta\), then we can write

\[y = tv + (1 - t)w\]
\[y' = t'v' + (1 - t')w'\]

for \(t, t' < \delta\). Then

\[||y - y'|| \leq t||v|| + t||w|| + t'||v'|| + t'||w'|| + ||w - w'|| \leq 4tC + \frac{2}{3} \epsilon \leq 4\delta C + \frac{2}{3} \epsilon,\]

where the bound on \(||w - w'||\) comes from the observation that \(K_2 \subseteq y_i + B\) has diameter \(\frac{2}{3} \epsilon\). Choosing \(\delta < \frac{\epsilon}{12C}\) will achieve the desired result.

(ii) The set \(K(\delta)\) is distinct from \(K\) if \(\delta\) is positive. Since \(U\) is a nonempty open subset of \(E\), it contains some element \(y \in E\). We claim that \(y \notin K(\delta)\): that is, we cannot write \(y = tv + (1 - t)w\) where \(t \leq 1 - \delta\), \(v \in K_1\), and \(w \in K_2\). Since \(y\) is an extreme point of \(K\), it will suffice to show that \(y \notin K_1\).

Since the weak topology on \(M\) is locally convex, we can choose a (weakly) open convex set \(V \subseteq M\) whose (weak) closure \(\overline{V}\) satisfies \((y - \overline{V}) \cap \overline{E} \subseteq U\). Since \(E - U\) is compact, it admits a finite covering by weakly open sets \(z_1 + V, z_2 + V, \ldots, z_k + V\) where \(z_i \in \overline{E}\). It follows that \(K_1\) is contained in the closed convex hull of \(\bigcup((z_i + V) \cap \overline{E})\), which is contained in the convex join of the sets \((z_i + \overline{V}) \cap K\). If \(y \in K_1\), then since \(y\) is an extreme point of \(K\), we deduce that \(y \in z_i + \overline{V}\) for some \(i\). Then \(z_i \in (y - \overline{V}) \cap \overline{E} \subseteq U\), contradicting our assumption that \(z_i \in \overline{E} - U\).