Let $A$ be a factor. In the last lecture, we associated to $A$ a linearly ordered abelian group $R(A)$: the collection $R(A)_+$ of nonnegative elements of $R(A)$ can be identified with the set of isomorphism classes of finite representations of $A$. Our first goal in this lecture is to determine the possible structures on $R(A)$. There are three cases to consider:

(a) The group $R(A)$ is trivial: that is, $A$ has no nontrivial finite representations. In this case, we say that $R(A)$ is a type III factor.

(b) There exists a smallest positive element of $R(A)$. This element corresponds to a representation $V$. Let $W \subseteq V$ be a proper $A$-submodule. Since $V$ is finite, we must have $W < V$. Since $V$ is a least positive element of $R(A)$, we have $W \cong 0$. This proves that $V$ is irreducible, so that the von Neumann algebra $A$ is type I.

(c) Suppose that $R(A)$ is nontrivial, but has no least positive element. Fix a positive element $V \in R(A).$ We define a map $\phi : R(A) \to \mathbb{R}$ as follows. Given $W \in R(A)$, let

$$Q_{\leq W} = \{ \frac{p}{q} : (q > 0) \land pV \leq qW \}$$

$$Q_{< W} = \{ \frac{p}{q} : (q > 0) \land pV < qW \}.$$

Since $V$ is positive, the Archimedean property of $R(A)$ implies that

$$-nV < W < nV$$

for $n$ sufficiently large. It follows that the sets $Q_{< W}$ and $Q_{\leq W}$ are nonempty and bounded above. Since they differ by at most a single rational number, they have the same supremum, which we will denote by $\phi(W)$. We have

$$-\phi(W) = -\sup Q_{\leq W} = \inf(-1) Q_{\leq W} = \inf(Q / Q_{< -W}) = \sup Q_{< -W} = \phi(-W).$$

It is clear that $\phi$ is monotone: if $W \leq W'$, then $Q_{\leq W} \subseteq Q_{\leq W'}$ so that $\phi(W) \leq \phi(W')$. We next show that $\phi$ is a group homomorphism. Let $W, W' \in R(A)$. If $\frac{p}{q} \in Q_{\leq W}$ and $\frac{p'}{q'} \in Q_{\leq W'}$, then we have

$$pV \leq qW \quad p'V \leq q'W'$$

so

$$pq'V \leq qq'W \quad p'qV \leq qq'W'$$

$$(pq' + p'q)V \leq qq'(W + W')$$
\[ \frac{p}{q} + \frac{p'}{q'} \in Q_{\leq W + W'}. \]

This proves that \( Q_{\leq W} \oplus Q_{\leq W'} \subseteq Q_{\leq W+W'} \) so that \( \phi(W) + \phi(W') \leq \phi(W+W') \). The reverse inequality then follows by applying the same arguments to \(-W\) and \(-W'\).

We now claim that \( \phi \) is injective. Assume otherwise; then there exists a positive element \( W \in R(A) \) such that \( \phi(W) = 0 \). Using the Archimedean property, we deduce that there exists an integer \( n \) such that \( V < nW \). Then \( \frac{1}{n} \in Q_W \), contradicting the assumption that \( \phi(W) = 0 \).

It remains to prove that \( \phi \) is surjective. Let us denote the image of \( \phi \) by \( K \subseteq \mathbb{R} \). We wish to show that \( K = \mathbb{R} \). Since \( K \) is a nontrivial subgroup of \( \mathbb{R} \) with no least element, it is dense in \( \mathbb{R} \). It will therefore suffice to show that \( K \) is closed. Let \( t \in K \); we wish to show that \( x \in K \). We can write \( t \) as the limit of a sequence of elements \( t_0 = t_1, t_2, \ldots \in K \) which is either increasing or decreasing; we will assume without loss of generality that the sequence is increasing. Then we can write \( t_{i+1} - t_i = \phi(W_i) \) for some finite representations \( W_i \) of \( A \). We will show that \( W = \bigoplus W_i \) is a finite representation of \( A \) and that \( x = t_0 + \phi(W) \) belongs to \( K \). To prove the second claim, it will suffice to show that \( t_0 + \phi(W) \geq r \) for every element \( r \in K \) such that \( r \geq x \). Writing \( r - t_0 = \phi(U) \), we are reduced to proving that \( W \leq U \) (which simultaneously proves the finiteness of \( W \)).

Note that we have \( \sum \phi(W_i) \leq \phi(U) \). In particular \( \phi(W_0) \leq \phi(U) \), so there exists an embedding \( f_0 : W_0 \hookrightarrow U \). Denote its orthogonal complement by \( U_1 \); then \( \phi(W_0) + \phi(W_1) \leq \phi(U) \) implies that \( W_1 \leq U_1 \) so we can choose an embedding \( f_1 : W_1 \hookrightarrow U_1 \subseteq U \). Proceeding in this way, we obtain a collection of embeddings \( f_i : W_i \hookrightarrow U \) with mutually disjoint images, which gives an isometric embedding \( \bigoplus W_i \hookrightarrow U \).

We say that a factor \( A \) is type II if the third case occurs: that is, if \( A \) has finite representations but no irreducible representations.

**Definition 1.** Let \( A \subseteq B(V) \) be a von Neumann algebra with commutant \( A' \). We will say that \( A \) is finite if \( V \) is finite when regarded as an \( A' \)-module.

**Remark 2.** In the situation of Definition 1, there is a bijective correspondence between closed \( A' \)-submodules of \( V \) and projections in \( A \). Moreover, if \( e \in A \) is a projection, then an isomorphism of \( V \) with \( eV \) (as \( A' \)-modules) can be identified with an operator \( u \in A \) satisfying \( uu^* = e, u^*u = 1 \). Note that the second condition implies that

\[ (uu^*)(uu^*) = uu^*, \]

so that \( uu^* \) is automatically a projection. It follows that \( A \) is finite if and only if the following condition is satisfied:

\[ (*) \quad \text{For every element } u \in A \text{ satisfying } u^*u = 1, \text{ we have } u^*u = 1. \]

In particular, this condition is intrinsic to \( A \): it does not depend on the embedding \( A \subseteq B(V) \).

We now study a mechanism for proving that a von Neumann algebra is finite.

**Proposition 3.** Let \( A \) be a von Neumann algebra and let \( \phi : A \to \mathbb{C} \) be a state. The following conditions are equivalent:

1. For every \( x, y \in A \), we have \( \phi(xy) = \phi(yx) \).
2. For every Hermitian element \( h \in A \) and every unitary element \( u \in A \), we have \( \phi(uhu^{-1}) = \phi(h) \).

**Proof.** To show that (1) \( \Rightarrow \) (2), take \( x = uh \) and \( y = u^{-1} \). For the converse, suppose that (2) is satisfied. Then every element \( h \in A \) satisfies \( \phi(uhu^{-1}) = \phi(h) \) (since the Hermitian elements generate \( A \) as a \( \mathbb{C} \)-vector space). Taking \( h = xu \), we obtain \( \phi(ux) = \phi(xu) \) for each \( x \in A \) and each unitary element \( u \in A \). To prove (1), it suffices to show that \( A \) is the \( \mathbb{C} \)-linear span of its unitary elements. It suffices to prove that every
Hermitian element \( y \in A \) belongs to this span. Replacing \( A \) by the abelian von Neumann algebra generated by \( y \), we can reduce to the case where \( A = L^\infty(X) \), in which case the desired result follows from elementary considerations.

**Definition 4.** Let \( A \) be a von Neumann algebra and let \( \phi : A \to \mathbb{C} \) be a state. We say that \( \phi \) is tracial if it satisfies the equivalent conditions of 3. In this case, we also say that \( \phi \) is a finite trace. We say that \( \phi \) is faithful if, for every positive element \( x \in A \), either \( x = 0 \) or \( \phi(x) > 0 \).

**Proposition 5.** Let \( A \) be a von Neumann algebra. If \( A \) admits a faithful finite trace, then \( A \) is finite.

**Proof.** Let \( u \in A \) be a partial isometry satisfying \( u^*u = 1 \); we wish to show that \( uu^* = 1 \). Write \( e = uu^* \). Then \( e \) is a projection, and we have \( \phi(e) = \phi(uu^*) = \phi(u^*u) = \phi(1) \). Thus \( \phi(1 - e) = 0 \). Since \( 1 - e \) is positive and \( \phi \) is faithful, this implies that \( 1 - e = 0 \), so that \( e = uu^* = 1 \) as desired.

We have the following converse:

**Theorem 6.** Let \( A \) be a finite von Neumann algebra. Then \( A \) can be written as a (von Neumann algebra) product \( \prod A_\alpha \), where each \( A_\alpha \) admits a faithful finite trace which is ultraweakly continuous.

**Remark 7.** From the characterization given in Remark 2, it is easy to see that a product of finite von Neumann algebras is itself finite. Thus the criterion of Theorem 6 is both necessary and sufficient.

**Remark 8.** If \( A \) is a factor, then one can prove that every tracial state is automatically ultraweakly continuous. We will not use this fact.

Here is a rough idea of why Theorem 6 should be true. Assume for simplicity that \( A \subseteq B(V) \) is a factor, so that \( V \) is finite when regarded as a representation of \( A' \). There is a unique order-preserving isomorphism \( \rho : R(A') \to \mathbb{R} \) such that \( \rho(V) = 1 \). We can think of \( \rho \) as a function which assigns a “dimension” to each finite representation of \( A' \). In particular, if \( e \in A \) is a projection, then \( eV \) is a closed \( A' \)-submodule of \( V \), hence finite as a representation of \( A' \). It therefore has a well-defined dimension \( \rho(eA) \). We would like to define a tracial state \( \phi : A \to \mathbb{C} \) by the formula

\[
\phi(e) = \rho(eA).
\]

Unfortunately, this formula only makes sense when \( e \) is a projection: to get a state, we need to define \( \phi \) on arbitrary elements of \( A \). However, since \( A \) is generated by its projections, any (ultraweakly continuous) state \( \phi \) is determined by its restriction to the projections. We might then hope to show that the above prescription extends uniquely to a state \( \phi : A \to \mathbb{C} \). We postpone giving a real proof for the moment; we will return to the matter next week.

Let’s explore some of the consequences of having a faithful finite trace. Recall that for any state \( \phi : A \to \mathbb{C} \), we can associate an inner product on \( A \), given by \( (x,y) = \phi(y^*x) \). We then have

\[
(zx,y) = \phi(y^*zx) = \phi((z^*y)^*x) = (x,z^*y).
\]

In other words, the action of \( A \) on itself by left multiplication is a \(*\)-homomorphism. However, it is not at all obvious that the same is true for right multiplication: we have

\[
(xz,y) = \phi(y^*xz) \quad (x,yz^*) = \phi(zy^*x).
\]

To say that these expressions are the same (for all \( x, y, \) and \( z \)) is to say that the right action of \( A \) on itself is via \(*\)-homomorphisms. If we let \( V_0 \) denote the Hilbert space completion of \( A \) with respect to the inner product \( (,\)\), this implies that the right action of \( A \) on itself extends to a right action of \( A \) on \( V_0 \) (note that if \( z \in A \) has norm \( \leq 1 \), then we can write \( 1 = z^*z + z'^*z' \) for some \( z' \in A \). If we let \( r_z \) and \( r_{z'} \) denote right multiplication by \( z \) and \( z' \), we get \( r_{z'}^*r_z + r_z^*r_{z'} = 1 \), which forces \( r_z \) to have operator norm \( \leq 1 \).
Proposition 9. Let $A$ be a von Neumann algebra, let $\phi : A \to \mathbb{C}$ be a faithful finite trace which is ultraweakly continuous, and let $V_\phi$ denote the Hilbert space associated to $\phi$. The left action of $A$ on itself induces an embedding $\rho : A \hookrightarrow B(V_\phi)$. Let $A'$ denote its commutant. Then the right action of $A$ on $V_\phi$ induces an isomorphism $\rho' : A^{\text{op}} \to A'$.

We will give the proof of this (and deduce some consequences) in the next lecture.