Let us begin by setting up some terminological conventions which will we will use in this course. We will always work over the field \( \mathbb{C} \) of complex numbers. By an \textit{algebra} we will mean an associative ring \( A \) with a map \( \mathbb{C} \to \text{Z}(A) \) (where \( \text{Z}(A) \) is the center of \( A \)).

Let \( A \) be an algebra. A \textit{norm} on \( A \) is a function \( ||| \cdot ||| : A \to \mathbb{R}_{\geq 0} \) satisfying the following axioms:

\[
|||x||| = 0 \text{ if and only if } x = 0 \\
|||x + y||| \leq |||x||| + |||y||| \\
|||\lambda x||| = |\lambda|||x||| \text{ if } \lambda \in \mathbb{C} \\
|||xy||| \leq |||x||| \cdot |||y|||
\]

Every norm on \( A \) determines a metric \( d \), given by \( d(x, y) = |||x - y||| \). We will say that \( A \) is \textit{complete} if is complete with respect to this metric. A \textit{Banach algebra} is a complete normed algebra.

\textbf{Remark 1.} Unless otherwise specified, we will always assume that our algebras are equipped with a unit. We will encounter nonunital algebras as well. In this case, some of the above definitions need to be modified.

\textbf{Example 2.} Let \( X \) be a compact topological space. Then the algebra \( \mathcal{C}^0(X) \) of continuous functions on \( X \) is a Banach algebra.

\textbf{Example 3.} Let \( V \) be a Hilbert space (or, more generally, a Banach space) and let \( \mathcal{B}(V) \) be the algebra of bounded operators on \( V \). Then \( \mathcal{B}(V) \) is a Banach algebra with respect to the \textit{operator norm}, given by

\[
||f|| = \sup \{|f(v)||, 0\}_{v \in V; ||v|| = 1}
\]

\textbf{Example 4.} Let \( G \) be a locally compact group, and let \( L^1(G) \) denote the Banach space of integrable functions on \( G \) (with respect to Haar measure). The operation of convolution equips \( L^1(G) \) with the structure of a Banach algebra. This Banach algebra is not unital unless \( G \) is discrete. However, it can be embedded in the larger \textit{unital} Banach algebra \( M(G) \) of finite Borel measures on \( G \) (where multiplication is again given by convolution, and the identity element is given by the point measure supported at the identity of \( G \)).

\textbf{Proposition 5.} Let \( A \) be a Banach algebra. Then the collection of invertible elements of \( A \) is open.

\textit{Proof.} Let \( x \in A \) be invertible; we wish to show that \( x \) has an open neighborhood in \( A \) consisting of invertible elements. Multiplying by \( x^{-1} \) (which is a homeomorphism of \( A \) with itself), we can reduce to the case \( x = 1 \). In this case, we wish to show that \( 1 + y \) is invertible provided that the norm of \( y \) is sufficiently small. The inverse of \( 1 + y \) is given by the power series

\[
1 - y + y^2 - y^3 + y^4 - \cdots,
\]

which converges for \( ||y|| < 1 \). 

\qed
Definition 6. Let $A$ be a Banach algebra and let $x \in A$. The spectrum of $x$ is the set of complex numbers $\lambda$ such that $x - \lambda$ is not invertible. We will denote the spectrum of $x$ by $\sigma(x)$. It follows from Proposition 5 that $\sigma(x)$ is a closed subset of the complex numbers.

Proposition 7. Let $A$ be a nonzero Banach algebra and let $x \in A$. Then the spectrum $\sigma(x)$ is nonempty.

Proof. Suppose otherwise; then the difference $\lambda - x$ is invertible for each $\lambda \in \mathbb{C}$. Let $\phi : A \to \mathbb{C}$ be a continuous linear functional. One can show that the map $\lambda \mapsto \phi\left(\frac{1}{\lambda - x}\right)$ is a holomorphic function on the complex numbers, which is bounded (in fact, it vanishes at infinity). It follows from complex analysis that this function is constant. Since $\phi$ is arbitrary, we deduce that the function $\lambda \mapsto \lambda - x$ is constant. This is only possible if $A = 0$. \qed

Corollary 8 (Gelfand-Mazur). Let $A$ be a Banach division algebra. Then $A \simeq \mathbb{C}$.

Proof. Let $x \in A$. Since $A \neq 0$, the spectrum $\sigma(x)$ is nonzero. Choose $\lambda \in \sigma(x)$, so that $\lambda - x$ is not invertible. Since $A$ is a division algebra, we deduce that $\lambda - x = 0$, so that $x = \lambda$. It follows that every element of $A$ is a scalar. \qed

We can regard Corollary 8 as a Banach-algebraic analogue of Hilbert’s Nullstellensatz:

Corollary 9. Let $A$ be a commutative Banach algebra, and let $m$ be a maximal ideal in $A$. Then the quotient $A/m$ is isomorphic to $\mathbb{C}$.

Proof. Let $\overline{m}$ be the closure of $m$. Proposition 5 implies that there is an open neighborhood $U$ of the identity element $1 \in A$ consisting of invertible elements, which cannot intersect $m$. It follows that $1 \notin \overline{m}$, so that $\overline{m}$ is a proper ideal of $A$. The maximality of $m$ implies that $m = \overline{m}$, so that $m$ is closed. Then the quotient $A/m$ inherits a complete norm, and is therefore a Banach algebra in its own right. Since $m$ is maximal, $A/m$ is a field. Applying Corollary 8, we deduce that $A/m \simeq \mathbb{C}$. \qed

Note that if $A$ is a Banach algebra, the spectrum of an element $x \in A$ cannot contain any complex number $\lambda > ||x||$, since for $\lambda > ||x||$ the formal series

$$1 + \frac{x}{\lambda} + \frac{x^2}{\lambda^2} + \frac{x^3}{\lambda^3} + \cdots$$

will converge to an inverse of $1 - \frac{x}{\lambda}$. It follows that $\sigma(x)$ is a closed and bounded subset of $\mathbb{C}$, hence compact.

Definition 10. Let $A$ be a Banach algebra and let $x \in A$. The spectral radius of $x$ is given by

$$\rho(x) = \max\{|\lambda| : \lambda \in \sigma(x)\}.$$ 

Our first nontrivial result is the following:

Theorem 11 (Gelfand’s Spectral Radius Formula). Let $A$ be a Banach algebra and let $x \in A$. Then

$$\rho(x) = \limsup ||x^n||^{\frac{1}{n}}.$$ 

Remark 12. One can say more: for example, the sequence $||x^n||^{\frac{1}{n}}$ converges to $\rho(x)$. We will not need this.

Proof. It will suffice to show that for every positive real number $\epsilon$, we have

$$\epsilon < \rho(x) \iff \epsilon < \limsup ||x^n||^{\frac{1}{n}}.$$
Suppose first that \( \limsup \|x^n\|^{\frac{1}{n}} \leq \epsilon \). We wish to show that \( \rho(x) \leq \epsilon \); that is, that \( x - \lambda \) is invertible whenever \( |\lambda| > \epsilon \). Rescaling if necessary, we may assume that \( \lambda = 1 \) (so that \( \epsilon < 1 \)). Choose \( \epsilon < \delta < 1 \). Since \( \limsup \|x^n\|^{\frac{1}{n}} \leq \epsilon \), we conclude that \( \|x^n\|^{\frac{1}{n}} \leq \delta \) for almost every integer \( n \). It follows that the power series \( 1 + x + x^2 + \cdots \) converges absolutely in \( A \) to an inverse of \( 1 - x \).

Suppose that \( \limsup \|x^n\|^{\frac{1}{n}} > \epsilon \); we wish to prove that \( \rho(x) > \epsilon \). Rescaling if necessary, we may assume that \( \epsilon = 1 \). Choose \( \delta \) with \( 1 < \delta < \delta^2 < \limsup \|x^n\|^{\frac{1}{n}} \), so that \( \|x^n\| \geq \delta^{2^n} \) for infinitely many values of \( n \). It follows that the sequence \( 1, \frac{x}{\delta}, \frac{x^2}{\delta^2}, \ldots \) is unbounded in \( A \). Let \( A^\vee \) denote the dual space of \( A \), and think of the elements \( x^i \) as linear functionals on \( A^\vee \). Using the uniform boundedness principle, we deduce that there exists a continuous linear functional \( \phi \in A^\vee \) such that the sequence \( \phi(1), \phi(\frac{x}{\delta}), \phi(\frac{x^2}{\delta^2}), \ldots \) is unbounded. Consider the function \( \lambda \mapsto \phi(\frac{\lambda}{\lambda - 2}) \), which is well-defined an holomorphic for \( \lambda \notin \sigma(x) \). For \( |\lambda| \) large, this function is given by

\[
\phi(1) + \phi(\frac{x}{\delta}) + \phi(\frac{x^2}{\delta^2}) + \cdots.
\]

If \( \rho(x) \leq 1 \), then complex analysis implies that this series converges absolutely \( |\lambda| > 1 \). Since it does not converge absolutely for \( \lambda = \delta \), we obtain a contradiction.

Corollary 13. Let \( A \) be a Banach algebra and let \( \chi : A \rightarrow C \) be an algebra homomorphism. Then \( \chi \) has norm \( \leq 1 \). In particular, \( \chi \) is continuous.

Proof. Let \( x \in A \). We wish to show that \( |\chi(x)| \leq \|x\| \). Assume otherwise; then \( |\chi(x)| > \|x\| \geq \rho(x) \), so that \( \chi(x) - x \) is invertible in \( A \). This is a contradiction, since the image of \( \chi(x) - x \) in \( C \) is zero.

Definition 14. Let \( A \) be a commutative Banach algebra. The spectrum of \( A \) is the collection of algebra homomorphisms \( \chi : A \rightarrow C \) (automatically continuous, by Corollary 13). We denote the spectrum of \( A \) by \( \text{Spec} \ A \).

Remark 15. Let \( A \) be a commutative Banach algebra and let \( x \in A \). A complex number \( \lambda \) belongs to the spectrum \( \sigma(x) \) if and only if \( \lambda - x \) is not invertible: that is, if and only if \( \lambda - x \) generates a nontrivial ideal in \( A \). Since every nontrivial ideal in \( A \) is contained in a maximal ideal, we see that \( \lambda \in \sigma(x) \) if and only if \( \lambda - x \) is contained in a maximal ideal of \( A \), which is then the kernel of an algebra homomorphism \( \chi : A \rightarrow C \). We deduce that \( \sigma(x) = \{ \chi(x) : \chi \in \text{Spec} \ A \} \).

Let us regard \( \text{Spec} \ A \) as a subset of the product

\[
\prod_{x \in A} \sigma(x).
\]

It is easy to see that \( \text{Spec} \ A \) is closed (with respect to the product topology on \( \prod_{x \in A} \sigma(x) \)), and therefore inherits the structure of a compact Hausdorff space.

We now study algebras with a bit more structure.

Definition 16. A \( * \)-algebra is an algebra \( A \) equipped with a map \( x \mapsto x^* \) satisfying the following axioms:

\[
(xy)^* = y^* x^* \quad (x + y)^* = x^* + y^* \quad \lambda^* = \overline{\lambda} \text{ if } \lambda \in C \quad x^{**} = x
\]

A \( C^* \)-algebra is a Banach \( * \)-algebra whose norm satisfies the following identity:

\[
\|x\|^2 = \|x^* x\|
\]

Example 17. Let \( X \) be a compact Hausdorff space. Then \( C^0(X) \) is a \( C^* \)-algebra, with involution given by complex conjugation of functions.
**Example 18.** Let $V$ be a Hilbert space. Then the algebra of bounded operators $B(V)$ is a $*$-algebra, where we take $f^*$ to be the adjoint of $f$, characterized by the formula

$$(f^* v, w) = (v, f w).$$

In fact, it is a $C^*$-algebra. The inequality $||x^* x|| \leq ||x^*|| ||x|| = ||x|| ||x|| = ||x||^2$ is obvious. To prove the reverse inequality $||x||^2 \leq ||xx^*||$, it will suffice to show that for any positive real number $\epsilon < ||x||$, there exists a vector $v \in V$ with $||v|| = 1$ and $||x^* xv|| > \epsilon^2$. Using the definition of $||x||$, we can choose $v$ such that $||xv|| > \epsilon$, so that

$$\epsilon^2 > ||xv||^2 = (xv, xv) = (x^* xv, v) \leq ||x^* xv||.$$

**Example 19.** Let $G$ be a unimodular locally compact group and regard $L^1(G)$ as a nonunital Banach algebra via convolution. Then $L^1(G)$ is equipped with the structure of a $*$-algebra, given by

$$f^*(g) = \overline{f(g^{-1})}.$$  

It is generally not a $C^*$-algebra.

**Notation 20.** Let $A$ be a $*$-algebra. We say that an element $x \in A$ is Hermitian or self-adjoint if $x = x^*$. We say that $x$ is skew-Hermitian or skew-adjoint if $x^* = -x$. Every element $x \in A$ admits a unique decomposition $x = \Re(x) + \Im(x)$, where $\Re(x) = \frac{x + x^*}{2}$ is self-adjoint and $\Im(x) = \frac{x - x^*}{2}$ is skew-adjoint.

We say that an element $x \in A$ is normal if $x$ and $x^*$ commute; equivalently, $x$ is normal if $\Re(x)$ and $\Im(x)$ commute.

**Proposition 21.** Let $A$ be a $C^*$-algebra and let $x \in A$ be a normal element. Then $||x^n|| = ||x||^n$ for every positive integer $n$.

**Proof.** It will suffice to show that $||x^n||^2 = ||x||^{2n}$. Applying the $C^*$-identity, we can rewrite this as $||x^n x^n|| = ||x^* x^n||$. Since $x$ is normal, the left hand side can be rewritten $||x^n x^n||$. We may therefore replace $x$ by $x^* x$ and thereby reduce to the case where $x$ is Hermitian. In this case, the $C^*$-identity gives $||x^n|| = ||x||^n$. Iterating this argument, we obtain

$$||x^k|| = ||x||^k.$$

Choose an integer $m$ such that $m + n$ is a power of 2. We then have

$$||x^{m+n}|| = ||x^m x^n|| \leq ||x^m|| ||x^n|| \leq ||x||^m ||x||^n = ||x||^{m+n}.$$

Since equality holds, we must have equality throughout. Assuming $x \neq 0$, this gives

$$||x^m|| = ||x||^m ||x|| = ||x||^n.$$

**Corollary 22.** Let $A$ be a $C^*$ algebra and let $x \in A$ be a normal element. Then the spectral radius $\rho(x)$ coincides with the norm $||x||$.

**Proof.** Combine the spectral radius formula (Theorem 11) with Proposition 21.

For any commutative Banach algebra $A$, each element $x \in A$ determines a continuous map $\text{Spec} \ A \to \mathbf{C}$, given by $\chi \mapsto \chi(x)$. This map is an algebra homomorphism, called the Gelfand transform.

**Proposition 23.** Let $A$ be a commutative $C^*$-algebra. Then the Gelfand transform $u : A \to C^0(\text{Spec} \ A)$ is an isomorphism of $C^*$-algebras.
Proof. We first show that \( u \) is a map of \(*\)-algebras. Equivalently, we claim that every character \( \chi : A \to \mathbb{C} \) satisfies \( \chi(x^*) = \overline{\chi(x)} \). It will suffice to show that \( \chi \) carries Hermitian elements \( x \in A \) to real numbers. Define \( f : \mathbb{R} \to A \) by the formula
\[
f(t) = e^{itx} = \sum_{n} \frac{(itx)^{n}}{n!}.
\]
Then \( f \) satisfies \( f(t)^{-1} = f(-t) = f(t)^* \), so that the \( C^* \)-identity gives
\[
||f(t)||^2 = ||f(t)^* f(t)|| = 1.
\]
Since \( \chi \) is continuous and has norm \( \leq 1 \) (Corollary 13), we obtain
\[
1 \geq |\chi f(t)| = e^{it\chi(x)}.
\]
Since this is true for both positive and negative values of \( t \), we must have \( \chi(x) \in \mathbb{R} \).

We now note that the Gelfand transform \( u \) is isometric: for \( x \in A \) we have
\[
||u(x)|| = \sup \{|\chi \in \text{Spec } A : |\chi(x)|\} = \rho(x) = ||x||
\]
by Corollary 22. It follows that \( u \) is an isomorphism from \( A \) onto a closed \(*\)-subalgebra of \( C^0(\text{Spec } A) \). This subalgebra separates points: if \( \chi, \chi' \in \text{Spec } A \) are distinct, then we can choose \( x \in A \) such that \( \chi(x) \neq \chi'(x) \).

Applying the Stone-Weierstrass theorem, we deduce that the image of \( u \) is the whole of \( C^0(\text{Spec } A) \), so that \( u \) is an isomorphism.

**Corollary 24.** Every commutative \( C^* \)-algebra is isomorphic to \( C^0(X) \) for some compact Hausdorff space \( X \). Moreover, we can canonically recover \( X \) as the spectrum \( \text{Spec } A \).