In this lecture, we will complete our algebraic characterization of von Neumann algebra morphisms by proving the following result:

**Lemma 1.** Let $A$ be a von Neumann algebra and let $\mu : A \to \mathbb{C}$ be a linear functional. If $\mu$ is ultra-strongly continuous on the unit ball $A_{\leq 1}$, then $\mu$ is ultraweakly continuous.

In fact, we will prove the following:

**Proposition 2.** Let $A$ be a von Neumann algebra with unit ball $A_{\leq 1}$, and let $\mu : A \to \mathbb{C}$ be a linear functional. The following conditions are equivalent:

1. $\mu$ is ultraweakly continuous.
2. The kernel $\ker(\mu)$ is closed in the ultraweak topology.
3. $\mu$ is ultrastrongly continuous on $A_{\leq 1}$.
4. The set $\ker(\mu) \cap A_{\leq 1}$ is ultraweakly closed.
5. $\ker(\mu)$ is closed in the ultrastrong topology.
6. The set $\ker(\mu) \cap A_{\leq 1}$ is ultrastrongly closed.

We have an obvious web of implications

\[
\begin{array}{cccc}
(a) & (c) & (c') & (a') \\
& (d) & (d') & \\
(b) & & (b') & \\
& (a) & (c') & (d) \\
& & (d') & \\
\end{array}
\]

In particular, condition (a) is the strongest and condition (d') is the weakest. The results of the last lecture shows that a completely additive state satisfies (d'), and that a state which satisfies (a) is completely additive. We will prove Proposition 2 by showing that (d') $\Rightarrow$ (a). Actually, we will proceed by showing that

(d') $\Rightarrow$ (c') $\Rightarrow$ (a') $\Rightarrow$ (a).
The implications \((a') \Rightarrow (a)\) and \((b') \Rightarrow (b)\) are easy. If \(\ker(\mu)\) is closed (for whatever topology), then the quotient topology on \(A/\ker(\mu)\) is Hausdorff, and therefore agrees with the standard topology on \(A/\ker(\mu) \cong C\). It follows that the composite map \(A \to A/\ker(\mu) \to C\) is continuous.

**Lemma 3.** We have \((b) \Rightarrow (a)\). That is, every ultrastrongly continuous functional on a von Neumann algebra \(A \subseteq B(V)\) is ultraweakly continuous.

**Proof.** Let \(\mu : A \to B(V)\) be ultrastrongly continuous. Then there exists a vector \(v \in V^{\oplus \infty}\) such that \(|\mu(x)| \leq ||x(v)||\) for each \(x \in A\). Replacing \(V\) by \(V^\infty\), we can assume \(v \in V\). Define a functional \(\mu_0 : Av \to C\) by the formula \(\mu_0(x(v)) = \mu(x)\) (this is well-defined: if \(x(v) = y(v)\), then \((x-y)(v) = 0\), so that \(\mu(x-y) = 0\) and \(\mu(x) = \mu(y)\)). We have \(|\mu_0(x(v))| = |\mu(x)| \leq ||x(v)||\), so that \(\mu_0\) has operator norm \(\leq 1\). It follows that \(\mu_0\) extends to a continuous functional on the closure \(V_0 = \overline{Av} \subseteq V\). Since \(V_0\) is a Hilbert space, this functional is given by inner product with some vector \(w \in V_0\). Then

\[
\mu(x) = (x(v), w),
\]

so that \(\mu\) is ultraweakly continuous. \(\square\)

We will need the following basic result from the theory of convexity:

**Theorem 4.** Let \(W\) be a locally convex topological vector space (over the real numbers, say), and let \(K \subseteq W\). The following conditions are equivalent:

1. The set \(K\) is closed and convex.
2. There exists a collection of continuous functionals \(\lambda_\alpha : W \to \mathbb{R}\) and a collection of real numbers \(C_\alpha\) such that \(K = \{w \in W : (\forall \alpha)[\lambda_\alpha(w) \geq C_\alpha]\}\).

**Proof.** We may assume without loss of generality that \(K\) contains the origin. Let \(v \in W - K\). Since \(K\) is closed, there exists an open neighborhood \(U\) of the origin such that \((v + U) \cap K = \emptyset\). Since \(W\) is locally convex, we can assume that \(U\) is convex. Then \(K + U\) is a convex subset of the origin. For \(w \in W\), define

\[
||w|| = \inf\{t \in \mathbb{R}_{>0} : tw \in K + U\}.
\]

This is almost a prenorm on \(W\): the convexity of \(K + U\) gives

\[
||w + w'|| \leq ||w|| + ||w'||,
\]

and we obviously have

\[
||tw|| = t||w||
\]

for \(t \geq 0\). This generally does not hold for \(t < 0\): that is, we can have \(||w|| \neq ||-w||\). Note that \(||v|| \geq 1\) (since \(v \notin K + U\)). Define \(\mu : \mathbb{R}v \to \mathbb{R}\) by the formula \(\mu(tv) = t\), so that \(\mu\) satisfies the inequality \(\mu(w) \leq ||w||\) for \(w \in \mathbb{R} v\). The proof of the Hahn-Banach theorem allows us to extend \(\mu\) to a function on all of \(W\) satisfying the same condition. We have \(|\mu(w)| = \pm \mu(w) \leq 1\) for \(w \in U \cap -U\), so that \(\mu\) is continuous. Since \(\mu(v) = 1\), \(\mu\) does not vanish so there exists \(u \in U\) with \(\mu(u) = \epsilon > 0\). Then for \(k \in K\), we have \(\mu(k + u) \leq 1\), so that \(\mu(k) \leq 1 - \mu(u)\). Then

\[
\{w \in W : \mu(w) \leq 1 - \mu(u)\}
\]

is a closed half-space containing \(K\) which does not contain \(v\). \(\square\)

**Corollary 5.** Let \(A\) be a von Neumann algebra, and let \(K \subseteq A\) be a convex subset. Then \(K\) is closed for the ultraweak topology if and only if \(K\) is closed for the ultrastrong topology.

From Corollary 5 we get the implications \((b') \Rightarrow (a')\) and \((d') \Rightarrow (c')\). To complete the proof, it suffices to show that \((c') \Rightarrow (a')\). Recall that \(A\) admits a Banach space predual \(E\), and that the ultraweak topology on \(A\) coincides with the weak \(*\)-topology. The implication \((c') \Rightarrow (a')\) is a special case of the following more general assertion:
Theorem 6 (Kreĭn-Smulian). Let $E$ be a real Banach space and let $K \subseteq E^\vee$ be a convex set. For each real number $r \geq 0$, we let $E_{\leq r}$ denote the closed unit ball of radius $r$ in $E^\vee$. If each of the intersections $K_{\leq r} = K \cap E_{\leq r}$ is closed for the weak $*$-topology, then $K$ is closed for the weak $*$-topology.

Proof. Let $v \in E - K$; we wish to show that $v$ does not belong to the weak $*$-closure of $K$. Replacing $K$ by $K - v$, we can reduce to the case where $v = 0$. Since each $K_{\leq r}$ is closed in the weak $*$-topology, it is also closed in the norm topology. It follows that $K$ is closed in the norm topology. In particular, since $0 \notin K$, there exists a real number $\epsilon > 0$ such that $E_{\leq \epsilon}$ does not intersect $K$. By rescaling, we can assume that $\epsilon = 1$.

We construct a sequence of finite subsets $S_1, S_2, S_3, \ldots \subseteq E$ with the following properties:

(a) If $\mu \in K_{\leq n+1}$, then there exists $v \in S_1 \cup \cdots \cup S_n$ such that $\mu(v) > 1$.

(b) If $v \in S_n$, then $||v|| = \frac{1}{n}$.

Assume that $S_1, \ldots, S_{n-1}$ have been constructed, and set

$$K(n) = K_{\leq n+1} \cap \{ \mu \in E : (\forall v \in S_1 \cup \cdots \cup S_{n-1}) [\mu(v) \leq 1] \}$$

Then $K(n)$ is a weak $*$-closed subset of $E^\vee$ which is bounded in the norm topology, and is therefore weak $*$-compact. By construction, $K(n)$ does not intersect $K_{\leq n}$. It follows that if $\mu \in K(n)$, then $||\mu|| > n$. We may therefore choose a vector $v \in E$ with $||v|| = \frac{1}{n}$ such that $\mu$ belongs to the set $U_n = \{ \rho \in E^\vee : \rho(v) > 1 \}$ (which is open for the weak $*$-topology). Since $K(n)$ is compact, we can cover $K(n)$ by finitely many such open sets $U_{v_1}, U_{v_2}, \ldots, U_{v_m}$. We then take $S_n = \{ v_1, \ldots, v_m \}$.

Let $S = \bigcup S_i$. Then $S$ is a countable subset of $E$; we can enumerate its elements as $v_1, v_2, \ldots$ (if $S$ is finite, we can extend this sequence by adding a sequence of zeros at the end). By construction, this sequence converges to zero in the norm topology on $E$. Let $C^0(\mathbb{Z}_{>0})$ denote the Banach space consisting of continuous maps

$$\mathbb{Z}_{>0} \to \mathbb{R}$$

which vanish at infinity: that is, the Banach space of sequences $(\lambda_1, \lambda_2, \ldots)$ which converge to zero. We have a map

$$f : E^\vee \to C^0(\mathbb{Z}_{>0})$$

given by $\mu \mapsto (\mu(v_1), \mu(v_2), \cdots)$. The image $f(K)$ is a convex subset of $C^0(\mathbb{Z}_{>0})$. By construction, if $\mu \in K$ then $\mu(v_i) > 1$ for some $i$, so that $f(K)$ does not intersect the unit ball of $C^0(\mathbb{Z}_{>0})$. It follows that 0 does not belong to the closure of $f(K)$. Applying Theorem 4, we see that there is a continuous functional $\rho : C^0(\mathbb{Z}_{>0}) \to \mathbb{R}$ such that $\rho(K) \subseteq \mathbb{R}_{\geq 1}$. This functional is given by a summable sequence of real numbers $(c_1, c_2, \ldots)$, and satisfies

$$\sum c_i \mu(v_i) \geq 1$$

for each $\mu \in K$. Set $v = \sum c_i v_i$; then $v \in E$ is a vector satisfying $\mu(v) \geq 1$ for $\mu \in K$. Then $K$ is contained in the weak $*$-closed set $\{ \mu \in E^\vee : \mu(v) \geq 1 \}$, so that 0 does not belong to the closure of $K$. \hfill \Box