The following result provides an intrinsic characterization of von Neumann algebras:

**Theorem 1.** Let $A$ be a $C^*$-algebra. Suppose there exists a Banach space $E$ and a Banach space isomorphism $A \cong E^\vee$. Then there exists a von Neumann algebra $B$ and an isomorphism of $C^*$-algebras $A \to B$ (in other words, $A$ admits the structure of a von Neumann algebra).

We will prove Theorem 1 under the following additional assumption:

$($*)$ For each $a \in A$, the operations $l_a, r_a : A \to A$ given by left multiplication on $A$ are continuous with respect to the weak $\ast$-topology (arising from the identification $A \cong E^\vee$).

**Remark 2.** We have seen that every von Neumann algebra admits a Banach space predual, such that the weak $\ast$-topology coincides with the ultraweak topology. Since multiplication in a von Neumann algebra is separately continuous in each variable for the ultraweak topology, condition $($*)$ is satisfied in any von Neumann algebra.

Let us now explain the proof of Theorem 1. Fix an isomorphism $\phi : A \to E^\vee$. We can think of $\phi$ as giving a bilinear pairing between $A$ and $E$, which in turn determines a bounded operator $\phi' : E \to A^\vee$. Let $\hat{\phi} : A^\vee \to E^\vee$ denote the dual of $\phi'$. The map $\hat{\phi}$ is continuous with respect to the weak $\ast$-topologies on $A^\vee$ and $M^\vee$, respectively, and fits into a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & M^\vee \\
\downarrow{\rho} & & \downarrow{\phi} \\
A^\vee & \xrightarrow{\hat{\phi}} & A^\vee.
\end{array}
$$

Here $\rho$ is the canonical map from $A$ into its double dual. The map $\hat{\phi}$ is uniquely determined by these properties (since $A$ is dense in $A^\vee$ with respect to the weak $\ast$-topology). We have seen that $A^\vee$ admits the structure of a von Neumann algebra, and that $\rho$ can be considered as a $C^*$-algebra homomorphism which exhibits $A^\vee$ as the von Neumann algebra envelope of $A$. Let $r = \phi^{-1} \circ \hat{\phi}$. Then $r$ is a left inverse to the canonical inclusion $\rho : A \to A^\vee$.

Fix an element $a \in A$. Let $l_a : A \to A$ denote the operation given by left multiplication by $A$, and let $l_{\rho(a)} : A^\vee \to A^\vee$ be defined similarly. Consider the diagram

$$
\begin{array}{ccc}
A^\vee & \xrightarrow{r} & A \\
\downarrow{l_{\rho(a)}} & & \downarrow{l_a} \\
A^\vee & \xrightarrow{r} & A.
\end{array}
$$

Since $\rho$ is an algebra homomorphism, this diagram commutes on the subset $\rho(A) \subseteq A^\vee$. Using assumption $(*)$, we see that all of the maps in this diagram are continuous if we regard $A^\vee$ and $A \cong E^\vee$ as equipped
with the weak \*$*$-topologies. Since the image of \( \rho \) is weak \*$*$-dense in \( A^{\vee \vee} \), we conclude that the diagram commutes.

Let \( K \subseteq A^{\vee \vee} \) denote the kernel of \( r \). Since \( r \) is continuous with respect to the weak \*$*$-topologies, \( K \) is closed with respect to the weak \*$*$-topology on \( A^{\vee \vee} \) (which coincides with the ultraweak topology). If \( x \in K \), we have

\[
r(\rho(a)x) = ar(x) = 0
\]

so that \( \rho(a)x = K \). The set \( \{ y \in A^{\vee \vee} : yx \in K \} \) is ultraweakly closed (since multiplication by \( x \) is ultraweakly continuous) and contains the image of \( \rho \). Since \( \rho(A) \) is ultraweakly dense in \( A^{\vee \vee} \), we deduce that \( \{ y \in A^{\vee \vee} : yx \in K \} \) contains all of \( A^{\vee \vee} \). It follows that \( K \) is a left ideal in \( A^{\vee \vee} \).

The same argument shows that \( K \) is a right ideal in \( A^{\vee \vee} \). Since \( K \) is ultraweakly closed, the results of the last lecture show that \( K \) is a \*$*$-ideal, and that the von Neumann algebra \( A^{\vee \vee} \) decomposes as a product

\[
A^{\vee \vee} \simeq A^{\vee \vee}/K \times K.
\]

Set \( B = A^{\vee \vee}/K \). The composite map

\[
A \to A^{\vee \vee} \to B
\]

is a \( C^* \)-algebra homomorphism and an isomorphism on the level of vector spaces, hence an isomorphism of \( C^* \)-algebras. This completes the proof of Theorem 1 (under the additional assumption \((*)\)).

In fact, we can say a bit more. Let us regard \( E \) as a subspace of its double dual \( E^{\vee \vee} \simeq A^{\vee} \), so that every vector \( e \) determines a functional \( \mu_e : A \to C \). Every such functional extends to a weak \*$*$-continuous map \( \hat{\mu}_e : A^{\vee \vee} \to C \), given by the composition

\[
A^{\vee \vee} \xrightarrow{\phi} E^{\vee} \xrightarrow{\xi} C.
\]

This composite map is ultraweakly continuous (since the weak \*$*$-topology on \( A^{\vee \vee} \) coincides with the ultraweak topology) and annihilates \( K \) (since \( K = \ker(r) = \ker(\hat{\phi}) \)). It follows that \( \hat{\mu}_e \) descends to an ultraweakly continuous functional \( B \to C \). In other words, the functional \( \mu_e \) is ultraweakly continuous if we regard \( A \) as a von Neumann algebra using the isomorphism \( A \cong B \).

Let \( F \subseteq A^{\vee} \) be the collection of ultraweakly continuous functionals with respect to our von Neumann algebra structure on \( A \), so that we can regard \( E \) as a closed subspace of \( F \). Consider the composite map

\[
A \to A^{\vee \vee} \to F^{\vee} \to E^{\vee}.
\]

Since \( A \) is a von Neumann algebra, the composition of the first two maps is an isomorphism. Since the composition of all three maps is an isomorphism by assumption, we conclude that the map \( F^{\vee} \to E^{\vee} \) is an isomorphism. This implies that \( E = F \): that is, \( E \) can be identified with the subspace of \( A^{\vee} \) consisting of \( \text{all} \) ultraweakly continuous functionals on \( A \). In particular, the weak \*$*$-topology on \( A \) agrees with the ultraweak topology given by the von Neumann algebra structure on \( A \).

It is natural to ask to what extent the Banach space \( E \) appearing in Theorem 1 is unique. Suppose we are given two Banach spaces \( E \) and \( E' \), together with isomorphisms

\[
E^{\vee} \simeq A \simeq E'^{\vee}.
\]

Can we then identify \( E \) with \( E' \)? In this situation, we can think of \( E \) and \( E' \) as subspaces of the dual space \( A^{\vee} \); we then ask: do these subspaces necessarily coincide? Our analysis shows that \( E \) determines an isomorphism of \( A \) with a von Neumann algebra \( B \), and that as a subspace of \( A^{\vee} \) we can identify \( E \) with those linear functionals which are ultraweakly continuous on \( B \). Similarly, \( E' \) determines an isomorphism \( A \simeq B' \). Asking if \( E = E' \) (as subspaces of \( A^{\vee} \)) is equivalent to asking if the \( C^* \)-algebra isomorphism \( B \simeq A \simeq B' \) carries ultraweakly continuous functionals on \( B \) to ultraweakly continuous functionals on \( B' \).

We can therefore phrase the question as follows:

**Question 3.** Let \( B \) and \( B' \) be von Neumann algebras, and let \( f : B \to B' \) be a \*$*$-algebra isomorphism. Is \( f \) necessarily an isomorphism of von Neumann algebras? That is, is \( f \) automatically continuous with respect to the ultraweak topologies?
Definition 4. Let $B$ be a von Neumann algebra. We say that an element $e \in B$ is a projection if $e$ is Hermitian and $e^2 = e$. Given a pair of projections $e$ and $e'$, we will write $e \leq e'$ if $ee' = e'e = e$. We say that $e$ and $e'$ are orthogonal if $ee' = e'e = 0$. In this case, $e + e'$ is also a projection, satisfying

$$e \leq e + e' \geq e'.$$

If $B$ is given as the set of bounded operators on some Hilbert space $V$, then an element $e \in B$ is a projection if and only if it is given by orthogonal projection onto some closed subspace $W \subseteq V$. Let us denote such a projection by $e_W$. Note that $e_W \leq e_{W'}$ if and only if $W \subseteq W'$, and that $e_W$ and $e_{W'}$ are orthogonal if and only if $W$ and $W'$ are orthogonal.

Suppose we are given a collection of mutually orthogonal projections $\{e_W^\alpha\}_{\alpha \in I}$ in $B$. Let $W$ be the closed subspace of $V$ generated by the subspaces $W^\alpha$. Then the collection of all finite sums $\sum_{\alpha \in I_0} e_{W^\alpha}$ converges to the projection $e_W$ in the ultraweak topology (in fact, it even converges in the ultrastrong topology). It follows that $e_W \in B$. We can characterize $e_W$ as the smallest projection satisfying $e_W \geq e_W^\alpha$ for every index $\alpha$.

Definition 5. Let $A$ and $B$ be von Neumann algebras, and let $\phi : A \to B$ be a $*$-algebra homomorphism. We will say that $\phi$ is completely additive if, for every family of mutually orthogonal projections $e^\alpha$ in $A$, we have

$$\phi(\sum e^\alpha) = \sum \phi(e^\alpha).$$

The notion of a completely additive $*$-algebra homomorphism is entirely algebraic, since the projection $\sum e^\alpha$ can be characterized as the least upper bound for the set of projections $\{e^\alpha\}$ in $A$. It is clear that any ultraweakly continuous $*$-algebra homomorphism is additive. In the next lecture, we will prove the following converse:

Theorem 6. Let $\phi : A \to B$ be a $*$-algebra homomorphism between von Neumann algebras. Then $\phi$ is ultraweakly continuous if and only if $\phi$ is completely additive.

Corollary 7. Any $*$-algebra isomorphism between von Neumann algebras is ultraweakly continuous. In particular, the ultraweak topology on a von Neumann algebra $A$ depends only on the underlying $*$-algebra of $A$. 

\[3\]