The Image of J (Lecture 35)

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The chromatic convergence theorem implies that the homotopy groups of the p-local sphere spectrum $S_{(p)}$ can be recovered as the inverse limit of the tower

$$\cdots \to \pi_* L_{E(2)} S \to \pi_* L_{E(1)} S \to \pi_* L_{E(0)} S.$$

The bottom of this tower is easy to understand: it is the rational sphere $S_{\mathbf{Q}}$, which is homotopy equivalent to the Eilenberg-MacLane spectrum $H\,\mathbf{Q}$. Our goal in this lecture is to understand the next step up in the tower, $L_{E(1)} S$. For simplicity, we will assume that $p > 2$.

Our first step is to describe the $K(1)$-local sphere. Our starting point is the following:

Lemma 1. *For each n, the spectrum $E(n)$ is $K(n)$-local.*

Proof. Recall that $E(n)$ is the even periodic Landweber exact spectrum associated to the Lubin-Tate ring $R = W(k)[[u_1, \ldots, u_{n-1}]]$ associated to a formal group of height n over a perfect field k of characteristic p.

Choose a cofiber sequence

$$X \to S_{(p)} \to L^t_{n-1} S_{(p)}$$

where X is a filtered colimit of p-local finite spectra DF_α of type $\geq n$. The dual DX is given by the homotopy inverse limit of a pro-spectrum $\{F_\alpha\}$. Taking MU-homology, we get a pro-system of $\pi_* \mathrm{MU}$-modules $\mathrm{MU}_*\{F_\alpha\}$; the theory of v_n-self maps shows that this pro-system can be identified with $\{\pi_* \mathrm{MU}/(v_0^N, v_1^N, \ldots, v_{n-1}^N\}_{N \geq 0}$. Since $E(n)$ is Landweber exact, we conclude that the pro-system $E(n)_*\{F_\alpha\}$ is equivalent to $\{\pi_* E(n)/(v_0^N, \ldots, v_{n-1}^N\}$. Since R is complete with respect to its maximal ideal, we conclude that the natural map $E(n) \to \varprojlim E(n)^{F_\alpha}$ is a homotopy equivalence. To prove that $E(n)$ is $K(n)$-local, it therefore suffices to show that each $E(n)^{F_\alpha}$ is $K(n)$-local. Let Y be a $K(n)$-acyclic spectrum; we wish to show that every map $Y \to E(n)^{F_\alpha}$ is nullhomotopic. This map is adjoint to a map $Y \otimes F_\alpha \to E(n)$. To show that such a map is nullhomotopic, it suffices to show that $E(n)_*(Y \otimes F_\alpha) \simeq 0$. This is equivalent to the statement that $K(m)_*(Y \otimes F_\alpha) \simeq 0$ for $m \leq n$. If $m < n$, this follows from the fact that F_α has type $\geq n$; if $m = n$, it follows from our assumption that Y is $K(n)$-acyclic. □

Let us now fix our notation a bit more precisely: choose a formal group f of height n over $\mathbf{F}_{p^n}$ such that all endomorphisms of f are defined over $\mathbf{F}_{p^n}$, and let $E(n)$ be the variant of Morava E-theory associated to this formal group. Then, in the homotopy category of spectra, $E(n)$ is acted on by a group G which fits into an exact sequence

$$0 \to \mathrm{End}(f)^\times \to G \to \mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p) \to 0.$$

In fact, the situation turns out to be even better than this: one can promote the "action of G on $E(n)$ up to homotopy" to a "homotopy coherent" action of G, which is continuous (with respect to the profinite topology on G). In this context, one can extract a continuous homotopy fixed point spectrum $E(n)^G$, which one can prove is equivalent to $L_{K(n)} S$.

All of this requires technology beyond the scope of this course. However, when $n = 1$ and p is odd, there is a lowbrow alternative. In this case, we can identify $E(n)$ with the p-adically completed K-theory spectrum $\widehat{K}$. The group G can be identified with the group $\mathbf{Z}_p^\times$ of p-adic units, which breaks up as a product

$$\mu_{p-1} \times (1 + p\mathbf{Z}_p)^\times$$

where the first factor is the finite group of $(p-1)$st roots of unity and the second is a pro-p group. When $p > 2$, the second group is actually the cyclic pro-p group: it is generated, for example, by the element $1 + p \in 1 + p\mathbf{Z}_p$.

Remark 2. It is easy to describe the induced action on $\pi_* \widehat{K}$. For any complex orientable cohomology theory E, we can identify $\pi_2 E$ with the dual of the Lie algebra of the associated formal group. Note that the action of $\mathbf{Z}_p^\times$ on $\widehat{K}$ is induced by its action on the multiplicative formal group $f(x,y) = x + y + xy$. The action of $\mathbf{Z}_p^\times$ on $\pi_2 \widehat{K}$ is therefore given by differentiating the action of $\mathbf{Z}_p^\times$ on the formal group itself: that is, it is given by the identity character of $\mathbf{Z}_p^\times$. Since $\pi_* \widehat{K} \simeq \mathbf{Z}_p[\beta^{\pm 1}]$ and $\mathbf{Z}_p^\times$ acts by ring homomorphisms, we conclude that $\mathbf{Z}_p^\times$ acts by the nth power of the identity character on $\pi_{2n} \mathbf{Z}_p$.

If $n \in \mathbf{Z}_p^\times$, we will denote the corresponding map $\widehat{K} \to \widehat{K}$ by ψ^n. One can show that these operations agree with the classical *Adams operations* in complex K-theory (which provides another proof of Remark 2).

If p is odd, then the group $\mathbf{Z}_p^\times$ is topologically cyclic: it has a generator given by $g = (\zeta, p+1)$, where ζ is any primitive $(p-1)$st root of unity. Consequently, we should expect taking continuous $\mathbf{Z}_p^\times$ homotopy fixed points to be easy: they should be given by the homotopy fiber of the map

$$\widehat{K} \stackrel{1-\psi^g}{\to} \widehat{K}.$$

Let us denote this homotopy fiber by F.

Proposition 3. *The map $\alpha : S \to F$ induces an isomorphism on $K(1)$-homology.*

Proof. Recall that $K(1)$ can be realized as a summand of $\widehat{K}/p$. It will therefore suffice to show that α induces an equivalence in $\widehat{K}/p$-homology. Since $\widehat{K}$ is Landweber exact, we have

$$\widehat{K}_0(\widehat{K}/p) \simeq \pi_0 \widehat{K} \otimes_L (\mathrm{MP}_0 \, \mathrm{MP}) \otimes_L \pi_0 \widehat{K}/p$$

(moreover, the homologies in all even degrees are the same by periodicity, and the homologies in odd degrees vanish). This is the $\mathbf{F}_p$-algebra which classifies isomorphisms of the multiplicative formal group with itself: that is, the algebra A of continuous $\mathbf{F}_p$-valued functions on the profinite group $\mathbf{Z}_p^\times$. In terms of this identification, the operation ψ^g is given by translation by g. We observe that $1 - \psi^g$ is a surjective map from A to itself, and its kernel is the one-dimensional $\mathbf{F}_p$-vector space of constant functions on $\mathbf{Z}_p^\times$. Using the long exact sequence

$$(\widehat{K}/p)_* F \to (\widehat{K}/p)_* \widehat{K} \stackrel{1-\psi^g}{\to} (\widehat{K}/p)_* \widehat{K},$$

we conclude that $(\widehat{K}/p)_* F \simeq \mathbf{F}_p[\beta^{\pm 1}] \simeq (\widehat{K}/p)_* S$. □

Since $\widehat{K}$ is $K(1)$-local, the spectrum F is also $K(1)$-local. It follows that:

Corollary 4. *The map $S \to F$ exhibits F as the $K(1)$-localization of S. In other words, the $K(1)$-local sphere $L_{K(1)} S$ is given by the homotopy fiber of the map $1 - \psi^g : \widehat{K} \to \widehat{K}$*

It follows that we have a long exact sequence

$$\pi_n \widehat{K} \stackrel{1-\psi^g}{\to} \pi_n \widehat{K} \to \pi_{n-1} L_{K(1)} S \to \pi_{n-1} \widehat{K}$$

which we can use to compute the homotopy groups of $L_{K(1)} S$. We note that ψ^g is the identity on $\pi_0 \widehat{K} \simeq \mathbf{Z}_p$, so that $1 - \psi^g$ vanishes on π_0 and we get isomorphisms

$$\pi_0 L_{K(1)} S \simeq \pi_{-1} L_{K(1)} S \simeq \mathbf{Z}_p.$$

The groups $\pi_n \widehat{K}$ vanish if n is odd. On $\pi_{2m} \widehat{K}$, the map $1 - \psi^g$ is given by $1 - g^m$ (Remark 2), and is therefore always injective for $m \neq 0$. Using the long exact sequence, we see that the even homotopy groups of $L_{K(1)} S$ vanish (except in degree zero), and we have an isomorphism $\pi_{2m-1} L_{K(1)} S \simeq \mathbf{Z}_p/(1 - g^m)$.

The cardinality of this group depends on m. If m is not divisible by $p-1$, then $g^m - 1$ is a unit modulo p so that $\pi_{2m-1}L_{K(1)}S$ vanishes. If $m = (p-1)m'$, then $g^m = (g^{p-1})^{m'}$ where g^{p-1} is a generator for the topologically cyclic pro-p-group $(1 + p\mathbf{Z}_p)^\times$. If we write $m' = p^k m''$, where m'' is prime to p, then g^m generates the cyclic subgroup $(1 + p^{k+1}\mathbf{Z}_p)^\times$, so that $1 - g^m$ is a generator for $p^{k+1}\mathbf{Z}_p \subseteq \mathbf{Z}_p$. We conclude:

Theorem 5. *The homotopy groups of $L_{K(1)}S$ are given as follows:*

$$\pi_n L_{K(1)}S \simeq \begin{cases} \mathbf{Z}_p & \text{if } n = 0, -1 \\ \mathbf{Z}/p^{k+1}\mathbf{Z} & \text{if } n+1 = (p-1)p^k m, m \not\equiv 0 \mod p \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 5 it is easy to describe the $E(1)$-local sphere. Recall that we have a homotopy pullback square

$$\begin{array}{ccc} L_{E(1)}S & \longrightarrow & L_{K(1)}S \\ \downarrow & & \downarrow \\ L_{E(0)}S & \longrightarrow & L_{E(0)}L_{K(1)}S. \end{array}$$

The localization $L_{E(0)}S$ is just the Eilenberg-MacLane spectrum $H\,\mathbf{Q}$. Theorem 5 implies that $\pi_n L_{E(0)}L_{K(1)}S \simeq \mathbf{Q}_p$ for $n = 0, 1$ and vanishes otherwise. Using the long exact sequence

$$\cdots \to \pi_{n+1}L_{E(0)}L_{K(1)}S \to \pi_n L_{E(1)}S \to \pi_n L_{K(1)}S \oplus \pi_n L_{E(0)}S \to\to \pi_n L_{E(0)}L_{K(1)}S \to \cdots,$$

we conclude that $\pi_n L_{E(1)}S \simeq \pi_n L_{K(1)}S$ unless $n \in \{0, -1, -2\}$. In these degrees, we have an exact sequence

$$0 \to \pi_0 L_{E(1)}S \to \mathbf{Z}_p \oplus \mathbf{Q} \to \mathbf{Q}_p \to \pi_{-1}L_{E(1)}S \to \mathbf{Z}_p \to \mathbf{Q}_p \to \pi_{-2}L_{E(1)}S \to 0.$$

Collecting these facts together, we obtain:

Theorem 6. *The homotopy groups of $L_{E(1)}S$ are given as follows:*

$$\pi_n L_{K(1)}S \simeq \begin{cases} \mathbf{Z} & \text{if } n = 0 \\ \mathbf{Q}_p/\mathbf{Z}_p & \text{if } n = -2 \\ \mathbf{Z}/p^{k+1}\mathbf{Z} & \text{if } n+1 = (p-1)p^k m, m \not\equiv 0 \mod p \\ 0 & \text{otherwise.} \end{cases}$$

There is an evident map $\pi_* S_{(p)} \to \pi_* L_{E(1)}S$, whose kernel is the second step in the chromatic filtration of $\pi_* S_{(p)}$. This map is obviously not surjective, since $\pi_* S_{(p)}$ is concentrated in positive degrees, while $\pi_* L_{E(1)}S$ is not. However, this turns out to be the only obstruction: the map $\pi_n S_{(p)} \to \pi_n L_{E(1)}S$ is surjective for $n \geq 0$. In other words, if $n > 0$, then every class in $\pi_n L_{E(1)}S \simeq \pi_n M_n(S)$ survives the chromatic spectral sequence. This is a result of Adams; let us briefly describe (without proof) the ideas involved.

Let $O(k)$ denote the orthogonal group of a k-dimensional vector space. Then $O(k)$ acts on the 1-point compactification of $\mathbb{R}^k$, fixing the point at infinity; this compactification can be identified with S^k. In particular, given a pointed map $X \to O(k)$ for any space X, we get a map $X \wedge S^k \to S^k$. Taking X to be a sphere, we get a map $\pi_n O(k) \to [S^{n+k}, S^k]$. Taking the limit as $k \mapsto \infty$, we get a homomorphism $\pi_n O \to \pi_n S$, where O denotes the infinite orthogonal group and S the sphere spectrum. This map is called the *J-homomorphism.*

The relationship between the J-homomorphism and the first chromatic layer can be stated as follows:

Theorem 7. *Let $\Im(J)_n$ denote the image of the J-homomorphism $\pi_n O \to \pi_n S \to \pi_n S_{(p)}$. For $n > 0$, the map $S_{(p)} \to L_{E(1)}S$ induces an isomorphism $\theta : \Im(J)_n \to \pi_n L_{E(1)}S$. In particular, the map $\pi_n S_{(p)} \to \pi_n L_{E(1)}S$ is surjective.*

The proof consists of two parts: proving that θ is injective and proving that θ is surjective. The surjectivity is not far from what we have done in class: we already know that each $\pi_n L_{E(1)} S$ is a cyclic group, so it suffices to show that θ hits a generator of the group; this can be proven by an explicit calculation.

Remark 8. The description of the image of the J-homomorphism was an important precursor to the development of the chromatic picture of homotopy theory: many of the ideas we have discussed had their origins in attempting to explain (and generalize) the "periodic behavior" exhibited by the image of the J homomorphism.