The Chromatic Convergence Theorem (Lecture 32)

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Fix a prime number $p$. For any $p$-local spectrum $X$, one can arrange its $E(n)$-localizations into the chromatic tower

$$
\cdots \rightarrow L_{E(2)}X \rightarrow L_{E(1)}X \rightarrow L_{E(0)}X.
$$

Our goal in this lecture and the next is to prove the following result:

**Theorem 1** (Chromatic Convergence). If $X$ is a finite $p$-local spectrum, then $X$ is a homotopy limit of its chromatic tower.

**Remark 2.** The collection of $p$-local spectra which satisfy the conclusion of Theorem 1 is obviously thick. It therefore suffices to prove Theorem 1 for a single $p$-local spectrum of type 0: for example, the $p$-local sphere.

For every spectrum $X$, let $C_n(X)$ denote the homotopy fiber of the map $X \rightarrow L_{E(n)}X$. Then $\lim C_n(X)$ is the homotopy fiber of the map $X \rightarrow \varprojlim L_{E(n)}X$. The chromatic convergence theorem is therefore equivalent to the following:

**Theorem 3.** The homotopy limit of the tower $\{C_n(S_p)\}$ is trivial. Even better: for every integer $m$, the tower of abelian groups $\{\pi_m C_n(S_p)\}$ is trivial (as a pro-abelian group).

The starting point for Theorem 3 is the following result, which we will prove in the next lecture:

**Proposition 4.** Each of the maps $C_n(S_p) \rightarrow C_{n-1}(S_p)$ induces the zero map $MU_*(C_n(S_p)) \rightarrow MU_*(C_{n-1}(S_p))$.

Let us assume Proposition 4 and see how it leads to a proof of Theorem 3. To this end, we recall the definition of the Adams-Novikov filtration on the homotopy groups $\pi_*X$ of a spectrum $X$. Let $I$ denote the homotopy fiber of the unit map $S \rightarrow MU$. There is an evident map $I \rightarrow S$, which induces a map $I^m \rightarrow S$ for each $m$. We say that an element $x \in \pi_*X$ has Adams-Novikov filtration $\geq m$ if $x$ lies in the image of the map $\pi_*(I^m \otimes X) \rightarrow \pi_*X$.

**Lemma 5.** Let $f : X \rightarrow Y$ be a map of spectra such that $f$ induces the zero map $\theta : MU_*(X) \rightarrow MU_*(Y)$. Then $f$ increases Adams-Novikov filtration. That is, if $x \in \pi_*X$ has Adams-Novikov filtration $\geq m$, then $f(x) \in \pi_*Y$ has Adams-Novikov filtration $\geq m + 1$.

**Proof.** Lift $x$ to a class $\overline{x} \in \pi_*(I^m \otimes X)$. We then obtain $f(\overline{x}) \in \pi_*(I^m \otimes Y)$ lifting $y$. To lift $y$ to $\pi_*(I^{m+1} \otimes Y)$, it suffices to show that the image of $\overline{y}$ vanishes in $I^m \otimes Y \otimes MU$. Consequently, it will suffice to show that $f$ induces the zero map

$$
\theta_m : MU_*(I^m \otimes X) \rightarrow MU_*(I^m \otimes Y).
$$

Recall that $MU_*(MU) \simeq (\pi_*MU)[b_1, b_2, \ldots]$ is a free $\pi_*$ $MU$-module on a basis consisting of monomials in the $b_i$. It follows that $MU_*(\Sigma I)$ is a free $\pi_*$ $MU$-module on a basis consisting of monomials of positive length in the $b_i$. In particular, $MU \otimes I$ is a free module over $MU$, so we have Kunneth decompositions

$$
MU_*(I^m \otimes X) = MU_*(I)^m \otimes_{\pi_*,MU} MU_*(X)
$$

$$
MU_*(I^m \otimes Y) = MU_*(I)^m \otimes_{\pi_*,MU} MU_*(Y)
$$

Since $\theta = 0$, it follows that $\theta_m = 0$. \qed
Combining Lemma 5 with Proposition 4, we deduce:

**Proposition 6.** For all \( m, n, \) and \( s \), the image of the map

\[
\pi_n C_{m+s} S(p) \rightarrow \pi_n C_{m} S(p)
\]

consists of elements having Adams-Novikov filtration \( \geq s \).

To complete the proof of Theorem 3, it will suffice to show the following:

**Proposition 7.** For every pair of integers \( m \) and \( n \), the Adams-Novikov filtration on \( \pi_n C_m (S(p)) \) is finite. That is, there exists an integer \( s \) such that every element \( x \in \pi_n C_m (S(p)) \) of Adams-Novikov filtration \( \geq s \) is trivial.

Let us now introduce some terminology which will be useful for proving Proposition 7.

**Definition 8.** Let \( f : X \rightarrow Y \) be a map of spectra. We say that \( f \) is phantom below dimension \( n \) if the following condition is satisfied: for every finite spectrum \( F \) of dimension \( \leq n \) and every map \( u : F \rightarrow X \), the composition \( f \circ u \) is nullhomotopic.

**Remark 9.** The map \( f \) is phantom if and only if it is phantom below dimension \( n \), for every integer \( n \).

**Definition 10.** A spectrum \( X \) is MU-convergent if, for every integer \( n \), there exists \( s \) such that the map \( I^s \otimes X \rightarrow X \) is phantom below dimension \( n \).

If \( X \) is MU-convergent and \( n, s \) are as in Definition 10, then the map \( I^s \otimes X \rightarrow X \) is trivial on \( \pi_n \) and so every element of \( \pi_n X \) having Adams-Novikov filtration \( \geq s \) is zero. Proposition 7 is therefore a consequence of the following:

**Proposition 11.** Let \( X \) be any connective spectrum. Then \( C_m (X) \) is MU-convergent for each \( m \geq 0 \).

We need a few preliminary observations.

**Lemma 12.** Let \( f : X \rightarrow Y \) phantom below dimension \( n \), and let \( W \) be a connective spectrum. Then the induced map \( X \otimes W \rightarrow Y \otimes W \) is phantom below dimension \( n \).

**Proof.** Let \( F \) be a finite spectrum of dimension \( \leq n \) and consider a map \( u : F \rightarrow X \otimes W \). We wish to prove that \( (f \otimes \text{id}_W) \circ u \) is nullhomotopic. We can write \( W \) as a filtered colimit of finite connective spectra \( W_\alpha \). Since \( F \) is finite, \( u \) factors through \( X \otimes W_\alpha \) for some \( \alpha \). Replacing \( W \) by \( W_\alpha \), we may assume that \( W \) is finite. In this case, we can identify \( u \) with a map \( v : DW \otimes F \rightarrow X \). Since \( W \) is connective, \( DW \otimes F \) has dimension \( \leq n \); it follows that \( f \circ v \) is nullhomotopic so that \( (f \otimes \text{id}_W) \circ u \) is nullhomotopic.

**Lemma 13.** Suppose we are given a fiber sequence of spectra

\[
X \rightarrow Y \rightarrow Z.
\]

If \( X \) and \( Z \) are MU-convergent, then \( Y \) is MU-convergent.

**Proof.** Fix an integer \( n \), and choose \( s \) such that the maps \( I^s \otimes X \rightarrow X \) and \( K^s \otimes Z \rightarrow Z \) are phantom below \( n \). We will show that the map \( I^{2s} \otimes Y \rightarrow Y \) is phantom below \( n \). Let \( F \) be a finite spectrum of dimension \( \leq n \) with a map \( u : F \rightarrow I^{2s} \otimes Y \). Since \( I^{2s} \otimes Z \rightarrow I^s \otimes Z \) is phantom below \( n \) (Lemma 12), the composite map

\[
F \rightarrow I^{2s} \otimes Y \rightarrow I^{2s} \otimes Z \rightarrow I^s \otimes Z
\]

is nullhomotopic. It follows that the composition

\[
F \otimes I^{2s} \otimes Y \rightarrow I^{s} \otimes Y
\]
factors through some map \( v : F \to I^\otimes s \otimes X \). Then the composition

\[
F \xrightarrow{u} I^\otimes 2s \otimes Y \to Y
\]

is given by

\[
F \xrightarrow{v} I^\otimes s \otimes X \to X \to Y
\]

and is therefore nullhomotopic.

**Lemma 14.** Let \( X \) be an MU-module spectrum. Then \( X \) is MU-convergent.

**Proof.** The unit map \( X \to MU \otimes X \) admits a section, given by the action of \( MU_p \) on \( X \). This is equivalent to the statement that the map \( I \otimes X \to X \) is nullhomotopic (and hence phantom below \( n \), for any \( n \)).  

**Lemma 15.** Let \( X \) be any spectrum. For each \( n \geq 0 \), the spectrum \( L_{E(n)} X \) is MU-convergent.

**Proof.** Let \( X^\bullet = E(n)^{\otimes (\bullet+1)} \otimes X \) and let \( \{ \text{Tot}^m X^\bullet \} \) be the \( E(n) \)-based Adams tower of \( X \). The proof of the smash product theorem shows that \( \{ \text{Tot}^m X^\bullet \} \) is equivalent to the constant tower with value \( L_{E(n)} X \). It follows that \( L_{E(n)} X \) is a retract of \( \text{Tot}^m X^\bullet \) for some \( m \). It therefore suffices to show that each \( \text{Tot}^m X^\bullet \) is MU-convergent. Each \( \text{Tot}^m X^\bullet \) is a finite homotopy inverse limit of the spectra \( X^k \); by Lemma 13 it suffices to show that each \( X^k \) is MU-convergent. But \( X^k \simeq E(n)^{\otimes k+1} \otimes X \) has the structure of an \( E(n) \)-module spectrum. Since \( E(n) \) is complex orientable, there is a map of ring spectra \( MU \to E(n) \) so that \( X^k \) admits an MU-module structure; the desired result now follows from Lemma 14.

**Lemma 16.** Let \( X \) be a connective spectrum. Then \( X \) is MU-convergent.

**Proof.** We claim that for any finite CW complex \( F \) of dimension \( \leq n \) and any map \( u : F \to I^{\otimes n+1} \otimes X \), the composite map \( u : F \to I^{\otimes n+1} \otimes X \to X \) is nullhomotopic. In fact, \( u \) itself is nullhomotopic, because \( I^{\otimes n+1} \otimes X \) is \( n \)-connected. To check this, we note that since \( X \) is connective it suffices to show that \( K \) is connected: that is, we have \( \pi_i K \simeq 0 \) for \( i \leq 0 \). This follows from the long exact sequence associated to the fiber sequence

\[
I \to S \to MU,
\]

since the map \( \pi_i S \to MU \) is bijective for \( i \leq 0 \) and surjective when \( i = 1 \).

**Proof of Proposition 11.** Let \( X \) be a connective spectrum. We have a fiber sequence

\[
C_n(X) \to X \to L_{E(n)} X
\]

where \( X \) is MU-convergent by Lemma 16 and \( L_{E(n)}(X) \) is MU-convergent by Lemma 15. It follows from Lemma 13 that \( C_n(X) \) is MU-convergent.