Localizations and the Adams-Novikov Spectral Sequence (Lecture 30)

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Throughout this lecture, we fix a ring spectrum \( E \). We will assume for simplicity that \( E \) is a structured ring spectrum. To any spectrum \( X \), we can associate the cosimplicial ring spectrum \( [n] \mapsto X \otimes E \otimes (E \otimes n + 1) \), which we will denote by \( X^\bullet \). The homotopy inverse limit of \( X^\bullet \) is called its totalization and denoted \( \text{Tot}(X^\bullet) \). It is given as an inverse limit of partial totalizations

\[
\cdots \rightarrow \text{Tot}^2(X^\bullet) \rightarrow \text{Tot}^1(X^\bullet) \rightarrow \text{Tot}^0(X^\bullet) \simeq X \otimes E,
\]
called the Adams tower for \( X \) with respect to \( E \). There is a canonical map \( X \rightarrow \text{Tot}(X^\bullet) \). We ask how closely this map approximates a homotopy equivalence.

The first observation is that \( X^\bullet \) depends only on the localization \( L_E X \): any \( E \)-homology equivalence \( X \rightarrow Y \) induces a homotopy equivalence of cosimplicial spectra \( X^\bullet \rightarrow Y^\bullet \). On the other hand, \( \text{Tot}(X^\bullet) \) is a homotopy inverse limit of \( E \)-modules, and is therefore automatically \( E \)-local. The best possible situation, then, is that \( \text{Tot}(X^\bullet) \) is an \( E \)-localization of \( X \): equivalently, the map \( X \rightarrow \text{Tot}(X^\bullet) \) induces an isomorphism in \( E \)-homology. This is equivalent to the assertion that \( E \otimes X \rightarrow E \otimes (\text{Tot}(X^\bullet)) \) is a homotopy equivalence.

The right hand side also admits a map to \( \text{Tot}(E \otimes X^\bullet) \). The augmented cosimplicial object \( [n] \mapsto E \otimes X \otimes (E \otimes (E \otimes n + 1)) \) is split: that is, it admits an extra codegeneracy map. It follows formally that the composite map

\[
E \otimes X \rightarrow E \otimes \text{Tot}(X^\bullet) \rightarrow \text{Tot}(E \otimes X^\bullet)
\]
is a homotopy equivalence. Consequently, we obtain the following:

**Proposition 1.** Let \( E \) be a structured ring spectrum and \( X \) a spectrum. Then the canonical map \( X \rightarrow \text{Tot}(X^\bullet) \) exhibits \( \text{Tot}(X^\bullet) \) as an \( E \)-localization of \( X \) if and only if \( E \otimes \text{Tot}(X^\bullet) \simeq \text{Tot}(E \otimes X^\bullet) \).

Note that \( \text{Tot}(E \otimes X^\bullet) \simeq \varprojlim \text{Tot}^n(E \otimes X^\bullet) \). Each partial totalization \( \text{Tot}^n \) is given by a finite homotopy inverse limit, and therefore commutes with smash products. It follows that \( \text{Tot}(E \otimes X^\bullet) \) can be identified with \( \varprojlim E \otimes \text{Tot}^n(X^\bullet) \). Consequently, the condition of Proposition 1 can be restated as follows: the canonical map

\[
E \otimes \varprojlim \text{Tot}^n(X^\bullet) \rightarrow \varprojlim E \otimes \text{Tot}^n(X^\bullet)
\]
is a homotopy equivalence.

To understand this condition better, it is convenient to work in the setting of pro-spectra. A pro-spectrum is a formal inverse limit of a filtered diagram of spectra (for our needs, it will be sufficient to consider inverse limits of towers). Morphism spaces are computed by the formula

\[
\text{Map}(\varprojlim \alpha X^\alpha, \varprojlim \beta Y^\beta) = \varprojlim \varprojlim \text{Map}(X^\alpha, Y^\beta).
\]
The collection of all pro-spectra form a homotopy theory, which we will denote by \( \text{Pro}(\text{Sp}) \). There is a forgetful functor \( U : \text{Pro}(\text{Sp}) \rightarrow \text{Sp} \), which carries a diagram \( \varprojlim \alpha X^\alpha \) to its homotopy inverse limit \( \varinjlim X^\alpha \). We say that a pro-spectrum \( \varprojlim \alpha X^\alpha \) is constant if, in \( \text{Pro}(\text{Sp}) \), it is homotopy equivalent to a constant tower

\[
\cdots \rightarrow X \rightarrow X.
\]
In this case, we have a canonical equivalence $\lim X_α \simeq X$.

If $\lim X_α''$ is a pro-spectrum and $E$ is any spectrum, then we can define a new pro-spectrum $E \otimes \lim X_α'' = \lim E \otimes X_α''$. We then have a natural map $E \otimes U(\lim X_α'') \to U(E \otimes \lim X_α'')$. This map is not always an equivalence, but it is obviously an equivalence when $\lim X_α''$ is constant. Applying this to our situation, we obtain the following:

**Proposition 2.** The equivalent conditions of Proposition 1 are satisfied whenever the tower

$$\cdots \to \operatorname{Tot}^2 X^* \to \operatorname{Tot}^1 X^* \to \operatorname{Tot}^0 X^*$$

is constant as a pro-spectrum.

Consequently, it is of interest for us to have a criterion for determining when a tower of spectra

$$\cdots \to Y(2) \to Y(1) \to Y(0)$$

is constant as a pro-spectrum. Recall that any such tower determines a spectral sequence $\{E_r^{p,q}, d_r\}$, which (in good cases) converges to $\pi_q \lim Y(n)$. Our goal is to establish the following criterion (a very imprecise version of a criterion of Bousfield):

**Proposition 3** (Bousfield). Let $\cdots \to Y(2) \to Y(1) \to Y(0)$ be a tower of spectra. Suppose that there exists an integer $s \geq 1$ with the following property: for every finite spectrum $F$, if $\{E_r^{p,q}, d_r\}$ is the spectral sequence associated to the tower

$$\cdots \to F \otimes Y(2) \to F \otimes Y(1) \to F \otimes Y(0),$$

then the groups $E_s^{p,q}$ vanish for $p \geq s$. Then the tower $\cdots \to Y(2) \to Y(1) \to Y(0)$ is constant as a pro-object.

To prove Proposition 3, we begin by fixing a tower of spectra

$$\cdots \to Y(2) \to Y(1) \to Y(0)$$

and assume that the associated spectral sequence $\{E_r^{p,q}\}$ satisfies $E_r^{p,q} \simeq 0$ for $p \geq s$. To exploit this hypothesis, we need to recall the details of the definition of the spectral sequence $\{E_r^{p,q}, d_r\}$. For $m \leq n$ let $F(m,n)$ denote the homotopy fiber of the map $Y(n) \to Y(m)$ (here we adopt the convention that $Y(m) \simeq 0$ for $m < 0$). Then $E_r^{p,q}$ is defined as the image of the map $\pi_q F(p + r - 1, p - 1) \to \pi_q F(p, p - r)$, and the differential $d_r$ carries $E_r^{p,q}$ into $E_r^{p+r,q-1}$. If $p < 0$, then $F(p, p - r)$ is contractible so that $E_r^{p,q}$ automatically vanishes. If $p \geq s$, then $E_r^{p,q}$ vanishes for $r \geq s$ by assumption. It follows that if $r \geq s$, then at least one of the groups $E_r^{p,q}$ and $E_r^{p+r,q-1}$ vanishes, so that the differential $d_r$ is identically zero. This proves:

(*) The groups $E_r^{p,q}$ are independent of $r$ for $r \geq s$. That is, the spectral sequence $\{E_r^{p,q}, d_r\}$ collapses at the $s$-page.

Now suppose $r > p$. Since $F(p, p - r) \simeq Y(p)$, we have $\pi_q F(p, p - r) \simeq \pi_q Y(p)$. In this case, $E_r^{p,q}$ is the image of the composite map

$$\pi_q F(p + r - 1, p - 1) \to \pi_q Y(p + r - 1) \to \pi_q Y(p).$$

The image of the first map is the kernel of the map $\pi_q Y(p + r - 1) \to \pi_q Y(p - 1)$. We therefore have:

($s'$) For $r > p$, the group $E_r^{p,q}$ is the intersection $\operatorname{Im}(\pi_q Y(p + r - 1) \to \pi_q Y(p)) \cap \ker(\pi_q Y(p) \to \pi_q Y(p - 1))$.

Combining (*) and ($s'$), we deduce:

($s''$) The intersection $\operatorname{Im}(\pi_q Y(p + r) \to \pi_q Y(p)) \cap \ker(\pi_q Y(p) \to \pi_q Y(p - 1)$ is independent of $r$, provided that $r \geq p, s$.
Lemma 4. For every integer \( k \geq 0 \), the intersection \( \text{Im}(\pi_q Y(p + r) \to \pi_q Y(p)) \cap \ker(\pi_q Y(p) \to \pi_q Y(p - k)) \) is independent of \( r \), provided that \( r \geq p, s \).

Proof. We use induction on \( k \). The case \( k = 0 \) is trivial, so assume that \( k > 0 \). Suppose that \( r \geq p, s \), and that \( x \in \pi_q Y(p + r) \) has trivial image in \( \pi_q Y(p - k) \). Let \( y \in \pi_q Y(p) \) be the image of \( x \); we wish to show that \( y \) lifts to \( \pi_q Y(p + r + 1) \). Let \( y' \) denote the image of \( y \) in \( \pi_q Y(p - 1) \). Then \( y' \) belongs to the kernel of the map \( \pi_q Y(p - 1) \to \pi_q Y(p - k) \). Since \( y' \) lifts to \( \pi_q Y(p + r) \), the inductive hypothesis implies that \( y' \) can be lifted to an element \( x' \in \pi_q Y(p + r + 1) \). Subtracting the image of \( x' \) from \( x \), we can reduce to the case \( y' = 0 \). Then \( y \in \ker(\pi_q Y(p) \to \pi_q Y(p - 1)) \), and the desired result follows from \((**)\).

Taking \( k = p + 1 \) in Lemma 4, we deduce that the image of the map \( \pi_q Y(p + r) \to \pi_q Y(p) \) is independent of \( r \), so long as \( r \geq p, s \). Let us denote this image by \( A(p)_* \). By construction, we have a sequence of surjections

\[ \cdots A(3)_* \to A(2)_* \to A(1)_* \to A(0)_* \]

By construction, each of these surjections fits into a short exact sequence

\[ 0 \to E^{p,*}_{\infty} \to A(p)_* \to A(p - 1)_* \to 0 \]

By assumption, the groups \( E^{p,*}_{\infty} \) vanish for \( p \geq s \). We deduce:

\((**)\) The maps \( A(p)_* \to A(p')_* \) are isomorphisms for \( p \geq p' \geq s \).

Let us now consider the tower of graded abelian groups

\[ \cdots \to \pi_* Y(4s) \xrightarrow{\theta_2} \pi_* Y(2s) \xrightarrow{\theta_1} \pi_* Y(s) \]

For \( m \geq 0 \), let \( K(m)_* \subseteq \pi_* Y(2^m s) \) be the kernel of the map \( \theta_m \). Note that \( K(m)_* \cap A(2^m s)_* = 0 \), since each \( \theta_m \) induces an isomorphism \( A(2^m s)_* \to A(2^{m-1} s)_* \). For any class \( x \in \pi_* Y(2^m s) \), the image \( \theta_m(x) \in A(2^{m-1} s)_* \), so that \( \theta_m(x) = \theta_m(x') \) for some \( x' \in A(2^m s)_* \). It follows that \( x = x' + x'' \), where \( x' \in A(2^m s)_* \) and \( x'' \in K(m)_* \). In other words, for \( m \geq 1 \) we have a direct sum decomposition

\[ \pi_* Y(2^m s) \cong A(2^m s)_* \oplus K(m)_* \]

It follows that, as a pro-object in graded abelian groups, the tower \( \{ \pi_* Y(2^m s) \} \) is equivalent to the constant group \( A(s)_* \).

Let \( Y = \lim Y(p) \cong \lim_m Y(2^m s) \). The Milnor exact sequence

\[ 0 \to \lim \pi_{* + 1} Y(p) \to \pi_* Y \to \lim \pi_* Y(p) \to 0 \]

gives \( \pi_* Y \cong A(s)_* \). For each integer \( p \geq 0 \), let \( Y(p) \to Y \) denote the cofiber of the canonical map \( Y \to Y(p) \). It follows that the maps \( \pi_* Y(2^m s) \to \pi_* Y(2^m s)/Y \) induce a composite isomorphism \( K(m)_* \subseteq \pi_* Y(2^m s) \to \pi_* Y(2^m s)/Y \).

We conclude that the tower of spectra

\[ \cdots \to Y(4s)/Y \to Y(2s)/Y \to Y(s)/Y \]

has the following property: each map in the tower is trivial on all homotopy groups.

Let us now return to the setting of Proposition 3: that is, we assume that the spectral sequence \( \{ E_{p,q}^r, d_r \} \) has vanishing \( E_{p,q}^r \) for \( p \geq s \) not only for the tower \( \{ Y(p) \} \), but also for \( \{ Y(p) \otimes F \} \) for every finite spectrum \( F \). The same reasoning shows that the maps

\[ \cdots \to (Y(4s)/Y)_* \to (Y(2s)/Y)_*, F \to (Y(s)/Y)_*, F \]

are zero. In other words, each of the maps \( Y(2^m s)/Y \to Y(2^m - 1 s)/Y \) is a phantom.
Lemma 5. A composition of two phantom maps is zero.

Proof. Fix a spectrum $X$, and consider a map $u : \bigoplus F_a \to X$, where the sum ranges over all homotopy equivalence classes of maps from finite spectra into $X$. Using the argument given in Lecture 17, we see that the homotopy fiber $X'$ of $u$ is equivalent to a retract of a sum of finite spectra. Now suppose we are given phantom maps $f : X \to Y$ and $g : Y \to Z$. Since $f$ is a phantom, $f \circ u \simeq 0$ and therefore $f$ is equivalent to a composition $X \to \Sigma X' \to Y$. Consequently, $g \circ f$ factors through the composition $\Sigma X' \to Y \to Z$. Since $g$ is a phantom and $\Sigma X'$ is a retract of a sum of finite spectra, the composition $g \circ v$ is nullhomotopic and therefore $g \circ f \simeq 0$. 

Applying this to our situation, we deduce that the maps

$$\cdots \to Y(16s)/Y \to Y(4s)/Y \to Y(s)/Y$$

are nullhomotopic, so that the pro-spectrum $\{Y(p)/Y\}$ is trivial. This proves that the tower $\{Y(p)\}$ is equivalent (as a pro-spectrum) to the constant spectrum $Y$. 
