Telescopic vs. $E_n$-Localization (Lecture 29)

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Let $p$ be a prime number, fixed throughout this lecture. Let $L$ be a Bousfield localization functor on $p$-local spectra. Our goal in this lecture is to obtain a structure theorem for $L$, under the assumption that $L$ is smashing.

Let us begin by fixing a bit of terminology. We say a spectrum $X$ is $L$-local if the map $X \to LX$ is an equivalence.

**Lemma 1.** Let $L$ be a localization functor. For $0 \leq n \leq \infty$, we have either $LK(n) \simeq 0$ or $LK(n) \simeq K(n)$.

**Proof.** We have a map of ring spectra $K(n) \to LK(n)$. Consequently, $LK(n)$ has the structure of a $K(n)$-module. If $LK(n) \neq 0$, then $LK(n)$ contains $K(n)$ (possibly shifted) as a retract. Since $LK(n)$ is $L$-local, we conclude that $K(n)$ is $L$-local so that $K(n) \simeq LK(n)$.

**Lemma 2.** Let $L$ be a smashing localization functor and let $E$ be a nonzero complex-oriented cohomology theory whose formal group has height exactly $n$. Then $LE \simeq 0$ if and only if $LK(n) \simeq 0$.

**Proof.** If $LE \simeq 0$, then $0 \simeq LK(n) \otimes LE \simeq LK(n) \otimes E$. Since $K(n) \otimes E \neq 0$, we conclude that $LK(n) \simeq 0$ (Lemma 1). Conversely, suppose that $LK(n) \simeq 0$. Then $0 \simeq LK(n) \otimes E \simeq K(n) \otimes LE$. On the other hand, $LE \otimes K(m) \simeq 0$ for $m \neq n$, since it is a complex oriented ring spectrum whose formal group has height exactly $m$ and exactly $n$. It follows from the nilpotence theorem that $LE \simeq 0$.

**Lemma 3.** Let $L$ be a smashing localization functor. If $LK(m) \simeq 0$, then $LK(n) \simeq 0$ for $n > m$.

**Proof.** For $k \geq 0$, let $M(k)$ denote the cofiber of the map $t_k : \Sigma^{2k} \MU(p) \to \MU(p)$, and let $R$ be the ring spectrum obtained by smashing (over $\MU(p)$) the spectra $\{M(k)\}_{k \neq p^{-1}, p^{-1}}$ with $\MU(p)[v_n^{-1}]$. For notational simplicity we will assume that $0 < m < n < \infty$, so that $\pi_* R \simeq F_p[v_m, v_n^{-1}]$. Note that $R[v_m^{-1}]$ is a ring spectrum whose associated formal group has height exactly $m$. It follows from Lemma 2 that $LR[v_m^{-1}] \simeq 0$. Since $L$ is smashing, we can identify $LR[v_m^{-1}]$ with the colimit of the sequence

$$LR \leftarrow \Sigma^{-2(p^m-1)}LR \leftarrow \Sigma^{-4(p^m-1)}LR \to \ldots$$

It follows that $1 \in \pi_0 LR$ vanishes in $\pi_0 \Sigma^{-2k(p^m-1)}R$ for $k \gg 0$; in other words, the image of $v_m^k$ vanishes in $\pi_* LR$. Let $R'$ denote the cofiber of the map $v_m^{k+1} : \Sigma^{2(k+1)(p^m-1)}R \to R$, so that $v_m^k$ vanishes in $\pi_* LR'$. Since $\pi_* R' \simeq F_p[v_m, v_n^{-1}]/(v_m^{k+1})$, we conclude that the map $\pi_* R' \to \pi_* LR'$ is not injective. In particular, $R'$ is not $L$-local. Note that $R'$ can be obtained as a successive extension of $k + 1$ copies of $R/v_m \simeq K(n)$. It follows that $K(n)$ is not $L$-local. According to Lemma 1, this means that $LK(n) \simeq 0$.

If $L$ is any localization functor, let us denote by $\ker(L)$ the collection of all $L$-acyclic spectra: that is, spectra $X$ such that $LX \simeq 0$.

**Lemma 4.** Let $L$ be a smashing localization functor, and let $n \geq 0$ be an integer. The following conditions are equivalent:

1. $LK(n) \simeq 0$. 

(2) $LK(m) \simeq 0$ for $n \leq m \leq \infty$.

(3) Every finite $p$-local spectrum $X$ of type $\geq n$ belongs to $\ker(L)$.

(4) There exists a finite $p$-local spectrum $X$ of type $n$ in $\ker(L)$.

**Proof.** The implication (1) $\Rightarrow$ (2) follows from Lemma 3. The implication (3) $\Rightarrow$ (4) is clear (since there exists a finite $p$-local spectrum of type $n$). To prove that (4) $\Rightarrow$ (1), we note that $\mathcal{L}X \simeq 0$ implies $\mathcal{L}X \otimes K(n) \simeq \mathcal{X} \otimes \mathcal{L}K(n) \simeq 0$. If $LK(n) \neq 0$, then $LK(n) \simeq K(n)$ so that $X \otimes \mathcal{L}K(n) \neq 0$, since $X$ has type $n$.

It remains to prove that (2) $\Rightarrow$ (3). Let $X$ be a $p$-local finite spectrum of type $\geq n$. We wish to prove that $\mathcal{L}X \simeq 0$. Let $R = X \otimes \mathcal{D}X$; since $\mathcal{L}X$ is an $LR$-module, it will suffice to show that $LR \simeq 0$. Since $LR$ is a ring spectrum, by the nilpotence theorem it will suffice to show that $LR \otimes K(m) \simeq 0$ for every $m$. If $m < n$, we have $LR \otimes K(m) \simeq LR \otimes K(m) \simeq 0$ since $R$ has type $\geq n > m$. If $m \geq n$, then $LR \otimes K(m) \simeq R \otimes LK(m) \simeq 0$ because $LK(m) \simeq 0$ by assumption (2).

\begin{itemize}
  \item[(A)] We have $LK(n) \simeq 0$ for all $0 \leq n < \infty$.
  \item[(B)] We have $LK(n) \simeq K(n)$ for all $0 \leq n < \infty$.
  \item[(C)] There exists an integer $n \geq 0$ such that $LK(n) \simeq K(n)$ but $LK(n+1) \simeq 0$.
\end{itemize}

In case (A), Lemma 2 guarantees that $L$ annihilates every finite $p$-local spectrum of type $\geq 0$. In particular, for every $X$ we have

$$\mathcal{L}X \simeq X \otimes LS(p) \simeq X \otimes 0 \simeq 0 :$$

that is, $L$ is the zero functor.

Let us now analyze case (C). Fix $n$ such that $LK(n) \simeq K(n)$ but $LK(n+1) \simeq 0$. Lemma 4 implies that $\ker(L)$ contains every finite spectrum of type $\geq n$. Conversely, if $X$ is a finite $p$-local spectrum such that $\mathcal{L}X \simeq 0$, we have

$$0 \simeq K(n) \otimes \mathcal{L}X \simeq LK(n) \otimes X \simeq K(n) \otimes X$$

so that $X$ must have type $\geq n$. In other words, the finite $p$-local spectra belonging to $\ker(f)$ are precisely the spectra of type $\geq n$: that is, the spectra which are $E(n)$-acyclic. Conversely, we have the following:

**Proposition 5.** Let $L$ be a smashing localization, and suppose that $LK(n) \simeq K(n)$. Then every spectrum which belongs to $\ker(L)$ is $E(n)$-acyclic.

**Remark 6.** An equivalent formulation is the following: if $L$ is a smashing localization with $LK(n) \simeq K(n)$, then every $E(n)$-local spectrum is $L$-local.

**Proof.** Let $X \in \ker(L)$. We wish to show that $X$ is $E(n)$-acyclic. Since $E(n)$ is Bousfield equivalent to $K(0) \oplus \cdots \oplus K(n)$, it suffices to show that $X$ is $K(m)$-acyclic for $m \leq n$. This follows from

$$K(m) \otimes X \simeq LK(m) \otimes X \simeq K(m) \otimes \mathcal{L}X \simeq 0,$$

since $L$ is smashing and $LK(m) \simeq K(m)$ for $m \leq n$ (Lemma 3).

Let us now return to case (C). If $L$ is a smashing localization with $LK(n) \simeq K(n)$ and $LK(n+1) \simeq 0$, then we conclude that $\ker(L)$ consists of $E(n)$-acyclic spectra, and contains all finite $E_n$-acyclic spectra. In other words, we have

$$\ker(L^t) \subseteq \ker(L) \subseteq \ker(L_{E(n)}).$$

The following conjecture of Ravenel is the main open problem left in the subject (though it is generally believed to be false):

**Conjecture 7** (Telescope Conjecture). The localization functors $L^t_n$ and $L_{E(n)}$ coincide. In particular, every smashing localization $L$ satisfying (C) above has the form $L^t_n$ for some $n \geq 0$. 

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It remains to treat the case (B): suppose that $L$ is a smashing localization with $LK(n) \simeq K(n)$ for $n \geq 0$. According to Remark 6, if $X$ is an $E(n)$-local spectrum for any $X$, then $X$ is $L$-local. In particular, the chromatic tower

$$\cdots \to LE(2)_pS \to LE(1)_pS \to LE(0)_pS$$

consists of $L$-local spectra, so that homotopy inverse limit of this tower is $L$-local. Next week we will prove the following:

**Theorem 8** (Chromatic Convergence Theorem). The homotopy inverse limit of the chromatic tower is $S_p$.

**Corollary 9.** Let $L$ be a smashing localization such that $LK(n) \simeq K(n)$ for $0 \leq n < \infty$. Then $L$ is equivalent to the identity functor.

**Proof.** Using the chromatic convergence theorem and Remark 6, we deduce that $S_p$ is $L$-local. Then, for any $p$-local spectrum $X$, we have

$$LX \simeq X \otimes LS_p \simeq X \otimes S_p \simeq X.$$

$\square$