Telescopic Localization (Lecture 28)

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Let p be a prime number, fixed throughout this lecture.

Let X be a p-local finite spectrum of type $\geq n$. In the last lecture, we saw that X admits a v_n-self map $f : \Sigma^k X \to X$. Moreover, such a map is asymptotically unique: if $f' : \Sigma^{k'} X \to X$ is another v_n-self map, then $f^i \simeq f'^j$ for some integers $i, j > 0$. It follows that the colimit of the sequence

$$X \xrightarrow{f} \Sigma^{-k} X \xrightarrow{f} \Sigma^{-2k} X \to \cdots$$

is independent of f. Let us denote this colimit by $X[f^{-1}]$.

We can describe $X[f^{-1}]$ more intrinsically as follows. Let $\mathcal{C}_{\geq n+1}$ denote the collection of all p-local finite spectra of type $> n$. Then $\mathcal{C}_{\geq n+1}$ determines a localization of the category of p-local spectra: that is, for every p-local spectrum X there is a canonical cofiber sequence

$$C(X) \to X \to L_n^t(X),$$

where $C(X)$ can be written as a filtered colimit of objects in $\mathcal{C}_{\geq n+1}$, and $L_n^t(X)$ is local with respect to $\mathcal{C}_{\geq n+1}$: in other words, if Y is a finite p-local spectrum of type $> n$, then every map $e : Y \to L_n^t(X)$ is nullhomotopic.

Proposition 1. *Let X be a finite p-local spectrum of type $\geq n$, and let f be a v_n-self map of X. Then $L_n^t(X) \simeq X[f^{-1}]$.*

More precisely, the canonical map $u : X \to X[f^{-1}]$ exhibits $X[f^{-1}]$ as a $\mathcal{C}_{\geq n+1}$-localization of X. To see this, we must verify two things:

(1) The fiber of the map $u : X \to X[f^{-1}]$ is a filtered colimit of objects of $\mathcal{C}_{\geq n+1}$. This is clear: the cofiber of u can be identified with the colimit of the sequence

$$0 \to \Sigma^{-k} X/X \to \Sigma^{-2k} X/X \to \ldots$$

Each $\Sigma^{-bk} X/X$ is (up to a shift) the cofiber of the v_n-self map f^b on X, which has type $> n$.

(2) The object $X[f^{-1}]$ is $\mathcal{C}_{\geq n+1}$-local. In other words, if Y is a finite spectrum of type $> n$, then every map $e : Y \to X[f^{-1}]$ is nullhomotopic. To see this, it suffices to show that $DY \otimes X[f^{-1}]$ is nullhomotopic. Without loss of generality, we may suppose that f induces the zero map on $K(m)_* X$ for $m \neq n$. It follows that $\mathrm{id}_{DY} \otimes f$ induces the zero map on $K(m)_*(DY \otimes X)$ for all integers m: here we use the assumption that Y is of type $> n$ and the Kunneth formula to see that $K(n)_*(DY \otimes X) \simeq 0$. By the nilpotence theorem, we conclude that $\mathrm{id}_{DY} \otimes f^a$ is nilpotent for $a \gg 0$. Replacing f by f^a, we may assume that $\mathrm{id}_{DY} \otimes f$ is nullhomotopic, so that $DY \otimes X[f^{-1}]$ is the colimit of a sequence of nullhomotopic maps

$$DY \otimes X \to DY \otimes \Sigma^{-k} X \to \cdots$$

and therefore contractible.

Remark 2. The functor L_n^t is sometimes referred to as *telescopic localization*. This is essentially a reference to Proposition 1, which gives an explicit construction of L_n^t (for type n-spectra) as a telescope: that is, as the homotopy colimit of a sequence of spectra.

We can view the theory of v_n-self maps as providing an explicit description of the effect of the localization functor L_n^t on finite p-local spectra of type $\geq n$. By applying this reasoning iteratively, we can understand L_n^t on arbitrary p-local finite spectra. To see this, let us begin with a p-local finite spectrum X. By convention, we can think of multiplication by p as a v_0-self map of X. That is, we can form the colimit $X[p^{-1}]$ of the sequence

$$X \xrightarrow{p} X \xrightarrow{p} X \xrightarrow{p} X \to \cdots$$

The above reasoning shows that $X[p^{-1}]$ can be identified with $L_0^t(X)$. We therefore have a cofiber sequence

$$\varinjlim_k \Sigma^{-1} X/p^k \to X \to L_0^t(X)$$

where X/p^k denotes the cofiber of multiplication by p^k on X. Applying the functor L_1^t, we get a commutative diagram

$$\begin{array}{ccccc} \varinjlim_k \Sigma^{-1} X/p^k & \longrightarrow & X & \longrightarrow & L_0^t(X) \\ \downarrow & & \downarrow & & \downarrow \\ L_1^t \varinjlim \Sigma^{-1} X/p^k & \longrightarrow & L_1^t(X) & \longrightarrow & L_1^t L_0^t(X) \end{array}$$

The vertical map on the right is an equivalence, since $L_0^t(X)$ is already local with respect to $\mathcal{C}_{\geq 2}$. It follows that the fiber F of the map $X \to L_1^t X$ can be identified with the fiber of the map

$$\varinjlim_k \Sigma^{-1} X/p^k \to L_1^t \varinjlim \Sigma^{-1} X/p^k$$

Since L_1^t is a smashing localization, it commutes with filtered colimits and we can therefore write F as the filtered colimit of the fibers of the maps

$$q : \Sigma^{-1} X/p^k \to \Sigma^{-1} L_1^t X/p^k.$$

Since each X/p^k is a finite p-local spectrum of type ≥ 1, Proposition **??** implies that $L_1^t X/p^k$ can be identified with $X/p^k[f_k^{-1}]$, where f_k is a v_1-self map of X/p^k. It follows that the fiber of q can be identified with the direct limit $\varinjlim \Sigma^{-2}(X/p^k)/f_k^l$. Thus F can be identified with the colimit $\varinjlim_k \varinjlim_l \Sigma^{-2}(X/p^k)/f_k^l$.

Here it is convenient to ignore the fact that f_k depends on k, and to denote all v_n-self maps by by the symbol v_n (so that $v_0 = p$). We can summarize our analysis informally as follows: we have a cofiber sequence

$$\varinjlim_{k_0, k_1} \Sigma^{-2} X/(v_0^{k_0}, v_1^{k_1}) \to X \to L_1^t(X).$$

This provides a somewhat explicit description of $L_1^t(X)$ as the cofiber of a map from a colimit of type ≥ 2-spectra into X.

Applying this argument repeatedly, we arrive at an "explicit" description of $L_n^t(X)$: it sits in a fiber sequence

$$\varinjlim_{k_0, \ldots, k_n} \Sigma^{-n} X/(v_0^{k_0}, \ldots, v_n^{k_n}) \to X \to L_n^t(X).$$

Since L_n^t is a smashing localization, it is in some sense determined by what it does to the (p-local) sphere spectrum. We have a cofiber sequence

$$\varinjlim_{k_0, \ldots, k_n} S^{-n}/(v_0^{k_0}, \ldots, v_n^{k_n}) \to S_{(p)} \to L_n^t S_{(p)}.$$

Smashing this cofiber sequence with X, we recover the sequence given above. However, there is another construction available in this context: instead of smashing with X, we can consider function spectra of maps into X. We get a fiber sequence

$$X^{L_n^t S_{(p)}} \to X \to \varprojlim X^{S^{-n}/(v_0^{k_0},\ldots,v_n^{k_n})}.$$

Unwinding the notation, we see that the function spectra on the right have a more direct description as the smash product of X with $S/(v_0^{k_0}, \ldots, v_n^{k_n})$, which we will denote by $X/(v_0^{k_0}, \ldots, v_n^{k_n})$. We can therefore think of the homotopy inverse limit on the right as a kind of completion of X.

Remark 3. Let $\mathcal{D}$ be the collection of all $\mathcal{C}_{\geq n+1}$-local spectra: that is, p-local spectra X such that every map $Y \to X$ is nullhomotopic if Y is a finite p-local spectrum of type $> n$. Then $\mathcal{D}$ is closed under shifts and homotopy colimits, and therefore determines another Bousfield localization functor R. That is, for every p-local spectrum X, there is a canonical cofiber sequence

$$D(X) \to X \to R(X)$$

where $D(X) \in \mathcal{D}$ and $R(X)$ is $\mathcal{D}$-local: that is, every map $g : Y \to R(X)$ is nullhomotopic if $Y \in \mathcal{D}$.

Proposition 4. *Let X be a p-local spectrum. Then the fiber sequence*

$$X^{L_n^t S_{(p)}} \to X \to \varprojlim X/(v_0^{k_0}, \ldots, v_n^{k_n})$$

can be identified with the fiber sequence of Remark 3.

In other words, the functor R of Remark 3 can be described as a "completion" with respect to the ideal $v_0, \ldots, v_n$, given by $X \mapsto \varprojlim X/(v_0^{k_0}, \ldots, v_n^{k_n})$.

As with Proposition 1, there are two things to prove:

(1) The function spectrum $X^{L_n^t S_{(p)}}$ belongs to $\mathcal{D}$. Let Y be a finite p-local spectrum of type $> n$; we wish to show that every map $u : Y \to X^{L_n^t S_{(p)}}$ is nullhomotopic. We can identify u with a map $Y \otimes L_n^t S_{(p)} \to X$. Such a map is automatically nullhomotopic, since $Y \otimes L_n^t S_{(p)} \simeq L_n^t Y$ vanishes by virtue of our assumption that Y has type $> n$.

(2) The homotopy inverse limit $\varprojlim X/(v_0^{k_0}, \ldots, v_n^{k_n})$ is $\mathcal{D}$-local. Since the collection of $\mathcal{D}$-local spectra is stable under homotopy inverse limits, it suffices to show that each term in the system is $\mathcal{D}$-local. Each of these terms has the form X^K, where K is a finite p-local spectrum of type $> n$. Let $Y \in \mathcal{D}$ and suppose we are given a map $u : Y \to X^K$; we wish to show that u is nullhomotopic. We can identify u with a map $Y \otimes K \to X$. To see that such a map is nullhomotopic, it suffices to show that $Y \otimes K \simeq 0$. This is clear, since $Y \in \mathcal{D}$ implies that $Y \simeq L_n^t Y$, so that

$$Y \otimes K \simeq L_n^t Y \otimes K \simeq Y \otimes L_n^t K \simeq 0,$$

by virtue of the fact that K has type $> n$.