Morava E-Theory and Morava K-Theory (Lecture 22)

April 27, 2010

Let k be a perfect field of characteristic p, and suppose we are given a formal group law f of height n over k. In the last lecture, we saw that the universal deformation of f is classified by the Lubin-Tate ring $R = W(k)[[v_1, \ldots, v_{n-1}]$. We note that this deformation of f over R is Landweber-exact: the sequence $v_0 = p, v_1, \ldots, v_{n-1}$ is regular by construction, and v_n has invertible image in $R/(v_0, v_1, \ldots, v_{n-1}) \simeq k$ by virtue of our assumption that the original formal group law f has height n.

Using results of previous lectures, we can construct an even periodic spectrum $E(n)$ with $\pi_* E(n) \simeq W(k)[[v_1, \ldots, v_{n-1}]][\beta^{\pm 1}]$, where β has degree 2. The cohomology theory $E(n)$ (which really depends not only on n, but on a choice of field k and a formal group of height n over k) is called *Morava E-theory*. It is also sometimes called *Lubin-Tate theory* or *completed Johnson-Wilson theory*.

Associated to $E(n)$ is a Bousfield localization functor $L_{E(n)}$. Note that a spectrum X satisfies $L_{E(n)} X = 0$ if and only if X is $E(n)$-acyclic: that is, the homology groups $E(n)_*(X) \simeq \mathrm{MP}_*(X) \otimes_L R$ vanish. We can associate to X a quasi-coherent sheaf $\mathcal{F}_X$ on $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$. The vanishing of $E(n)_*(X)$ is equivalent to the requirement that both $\mathcal{F}_X$ and $\mathcal{F}_{\Sigma(X)}$ (and therefore $\mathcal{F}_{\Sigma^k X}$ for every integer k) are supported on the closed substack $\mathcal{M}_{\mathrm{FG}}^{\geq n+1} \subseteq \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$. This is one sense in which $L_{E(n)}$ "behaves like" restriction to the open substack $\mathcal{M}_{\mathrm{FG}}^{\leq n} \subseteq \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbf{Z}_{(p)}$. This suggests that $L_{E(n)}$ should be a smashing localization. This is indeed the case:

Theorem 1 (Smash Product Theorem)**.** *The localization $L_{E(n)}$ is smashing: that is, it preserves direct sums.*

We will prove Theorem 1 later in this course.

Our next goal is to introduce a homotopy theoretic counterpart to the mechanism of restricting to the *closed* substack $\mathcal{M}_{\mathrm{FG}}^{n} \subseteq \mathcal{M}_{\mathrm{FG}}^{\leq n}$. This is more subtle, since $\mathcal{M}_{\mathrm{FG}}^{n}$ is not flat over $\mathcal{M}_{\mathrm{FG}}$, so we cannot proceed via Landweber's theorem.

Fix a prime p, and consider the p-local complex bordism spectrum $\mathrm{MU}_{(p)}$. This complex bordism spectrum has the structure of an E_∞*-ring*. In particular, there is a good theory of (structured) $\mathrm{MU}_{(p)}$-modules, and a relative smash product $(M, N) \mapsto M \otimes_{\mathrm{MU}_{(p)}} N$.

We have $\pi_* \mathrm{MU}_{(p)} \simeq L_{(p)} \simeq \mathbf{Z}_{(p)}[t_1, t_2, \ldots]$, where we may assume that $v_i = t^{p^i - 1}$ for each $i > 0$. By convention, we set $t_0 = p \in \pi_0 \mathrm{MU}_{(p)}$.

For each integer k, let $M(k)$ denote the cofiber of the map $\Sigma^{2k} \mathrm{MU}_{(p)} \rightarrow \mathrm{MU}_{(p)}$ given by multiplication by t_k.

Lemma 2. *Each $M(k)$ admits a unital and homotopy associative multiplication (in the category of $\mathrm{MU}_{(p)}$-module spectra).*

Proof. We fix the unit of $M(k)$ to be the evident map $u : \mathrm{MU}_{(p)} \rightarrow M(k)$. The smash product $M(k) \otimes_{\mathrm{MU}_{(p)}}$

$M(k)$ can be realized as the total homotopy cofiber of the commutative diagram

$$\begin{array}{ccc} \Sigma^{4k} \mathrm{MU}_{(p)} & \xrightarrow{t_k} & \Sigma^{2k} \mathrm{MU}_{(p)} \\ \downarrow{\scriptstyle t_k} & & \downarrow{\scriptstyle t_k} \\ \Sigma^{2k} \mathrm{MU}_{(p)} & \xrightarrow{t_k} & \mathrm{MU}_{(p)} . \end{array}$$

We let K denote the total cofiber of an analogous diagram, where we replace the upper left hand corner with the zero spectrum. In other words, K is the cofiber of the map

$$(\Sigma^{2k} M(k))^2 \stackrel{(t_k,t_k)}{\to} M(k).$$

There is an evident map $\alpha : K \to M(k)$. To define an $\mathrm{MU}_{(p)}$-linear multiplication on $M(k)$ (having u as a unit) is equivalent to factoring u as a composition

$$K \stackrel{\beta}{\to} M(k) \otimes_{\mathrm{MU}_{(p)}} M(k) \stackrel{\gamma}{\to} M(k)$$

in the setting of $\mathrm{MU}_{(p)}$-modules.

To produce such a factorization, it suffices to show that the composition

$$\ker(\beta) \to K \stackrel{\alpha}{\to} M(k)$$

is nullhomotopic. Note that $\ker(\beta)$ can be identified with the desuspension of the total cofiber of the square

$$\begin{array}{ccc} \Sigma^{4k} M(k) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0; \end{array}$$

that is, we have $\ker(\beta) \simeq \Sigma^{4k+1} M(k)$, and the relevant obstruction lives in $\pi_{4k+1} M(k) \simeq (L_p/t_k)_{4k+1} \simeq 0$.

We now show that the multiplication γ is homotopy associative (in the setting of $\mathrm{MU}_{(p)}$-modules). We have two natural multiplication maps

$$f, g : X = M(k) \otimes_{\mathrm{MU}_{(p)}} M(k) \otimes_{\mathrm{MU}_{(p)}} M(k) \to M(k).$$

We wish to prove that the difference $f - g$ is nullhomotopic. Note that X can be described as the total cofiber of a cube

$$\begin{array}{ccccccc} \Sigma^{6k} \mathrm{MU}_{(p)} & & \longrightarrow & & \Sigma^{4k} \mathrm{MU}_{(p)} & & \\ & \searrow & & & & \searrow & \\ & & \Sigma^{4k} \mathrm{MU}_{(p)} & & \longrightarrow & & \Sigma^{2k} \mathrm{MU}_{(p)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^{4k} \mathrm{MU}_{(p)} & & \longrightarrow & & \Sigma^{2k} \mathrm{MU}_{(p)} & & \\ & \searrow & & & & \searrow & \\ & & \Sigma^{2k} \mathrm{MU}_{(p)} & & \longrightarrow & & \mathrm{MU}_{(p)} . \end{array}$$

Let Y be the total cofiber of an analogous diagram obtained by replacing the upper left corner by zero. By construction, the difference $f - g$ is nullhomotopic on Y, so that $f - g$ factors as a composition

$$X \to X/Y \simeq \Sigma^{6k+3} \mathrm{MU}_{(p)} \to M(k).$$

Since $\pi_{6k+3} M(k) \simeq (L_{(p)}/t_k)_{6k+3} \simeq 0$, the second map is nullhomotopic so that $f \simeq g$ as desired. □

Remark 3. The multiplication on $M(k)$ constructed above is not unique: the same argument shows that the collection of such multiplications forms a torsor P for the group $\pi_{4k+2}M(k)$.

The torsor P has a canonical action of the permutation group Σ_2 (which acts on the smash product $M(k) \otimes_{\mathrm{MU}_{(p)}} M(k)$. If $p \neq 2$, then $\mathrm{H}^1(\Sigma_2; \pi_{2k+2}M(k)) \simeq 0$ so that P has a Σ_2-fixed point. This means that we can choose the multiplication on $M(k)$ to be homotopy commutative when $p \neq 2$.

Remark 4. By continuing the analysis of Lemma 2, one can show that $M(k)$ admits the structure of an A_∞-algebra over $\mathrm{MU}_{(p)}$.

Definition 5. Fix a prime number p and an integer $n > 0$. We let $K(n)$ denote the smash product (over $\mathrm{MU}_{(p)}$) of $\mathrm{MU}_{(p)}[v_n^{-1}]$ with $\bigotimes_{k \neq p^n - 1} M(k)$. The spectrum $K(n)$ is called *Morava K-theory*.

Using Lemma 2, we see that $K(n)$ has the structure of a homotopy associative $\mathrm{MU}_{(p)}$-algebra; if $p \neq 2$, we can even assume that K is homotopy commutative.

A simple calculation shows that the homotopy groups of $K(n)$ are given by

$$\pi_* K(n) \simeq (\pi_* \mathrm{MU}_{(p)})[v_n^{-1}]/(t_0, t_1, \ldots, t_{p^n-2}, t_{p^n}, \ldots) \simeq \mathbf{F}_p[v_n^{\pm 1}],$$

where v_n has degree $2(p^n - 1)$.

We have a map of ring spectrum $\mathrm{MU}_{(p)} \to K(n)$, giving a complex orientation on $K(n)$. This determines a formal group law over the ring $\pi_* K(n) \simeq \mathbf{F}_p[v_n^{\pm 1}]$, which has height exactly n.

Warning 6. When $p = 2$, the Morava K-theory spectra generally do not admit homotopy commutative ring structures. Nevertheless, the theory of complex orientations makes sense in this setting: though $K(n)$ itself is not homotopy commutative, the cohomology rings $K(n)^*(X)$ are commutative for many important spaces (like $\mathbf{C}\mathrm{P}^\infty$, $BU(n)$, and so forth) since they are given by $\mathrm{MU}^*(X) \otimes_L \mathbf{F}_p[v_n^{\pm 1}]$.

Warning 7. Our construction of the ring spectra $M(k)$ (and therefore the Morava K-theories $K(n)$) involve a number of arbitrary choices. We will later see that, as a spectrum, $K(n)$ does not depend on these choices.

We let $L_{K(n)}$ denote the localization with respect to the Morava K-theory $K(n)$. We will later see that $L_{K(n)}$ behaves like completion along the locally closed substack $\mathcal{M}_{\mathrm{FG}}^n \subseteq \mathcal{M}_{\mathrm{FG}}$.