The Proof of Quillen's Theorem (Lecture 10)

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At the end of the last lecture, we arrived at the following conclusion for the prime $p = 2$:

Proposition 1. *The second page of the mod p Adams spectral sequence for* MU *is given by*

$$E_2^{*,*} \simeq \mathbf{F}_p[b_i, \epsilon_j].$$

Here i ranges over nonnegative integers such that $i+1$ is not a power of p, and b_i has bidegree $(2i, 0)$, while ϵ_j is defined for all $j \geq 0$ and has bidegree $(p^j - 1, 1)$.

This calculation is valid for all p, not just the case $p = 2$. The proofs are essentially the same, but our use of algebraic geometry needs to be replaced by "super" algebraic geometry.

We define $c_i = \begin{cases} \epsilon_j & \text{if } i+1 = p^j \\ b_i & \text{otherwise.} \end{cases}$ so that we have an isomorphism $E_2^{*,*} \simeq \mathbf{F}_p[c_0, c_1, \ldots]$. Here each c_i has total degree $2i$. In particular, every nonzero element of $E_2^{*,*}$ has even total degree, so the Adams spectral sequence degenerates at the second page. We deduce the following:

Proposition 2. *The (mod p) Adams filtration on $\pi_* \mathrm{MU}$ has the following property:*

$(*)$ *The associated graded ring $\mathrm{gr}(\pi_* \mathrm{MU})$ is isomorphic to a polynomial algebra $\mathbf{F}_p[c_0, c_1, \ldots]$, with $c_i \in \mathrm{gr}_0(\pi_{2i} \mathrm{MU})$ for $i + 1 \neq p^j$, and $c_i \in \mathrm{gr}_1(\pi_{2i} \mathrm{MU})$ for $i + 1 = p^j$.*

Remark 3. In particular, the class c_0 can be lifted to an element of $F^1 \pi_0 \mathrm{MU}$, which is the kernel of the Hurewicz map

$$\pi_0 \mathrm{MU} \simeq \mathrm{H}_0(\mathrm{MU}; \mathbf{Z}) \simeq \mathbf{Z} \to \mathbf{F}_p \simeq \mathrm{H}_0(\mathrm{MU}; \mathbf{F}_p).$$

This map is nonzero modulo elements of Adams filtration 2, which includes the subgroup $p^2 \pi_0 \mathrm{MU} \simeq p^2 \mathbf{Z}$. It follows that (after modifying by a suitable scalar) we may assume that c_0 is represented by $p \in p\mathbf{Z} \simeq F^1 \pi_* \mathrm{MU}$.

Let R denote the polynomial ring $\mathbf{Z}[u_1, u_2, \ldots]$. We regard R as a graded ring where each class u_i has degree $2i$. We also regard R as filtered, where $F^i R$ is generated by monomials of the form $p^{m_1} u_1^{m_2} u_2^{m_3} \ldots$ for which $m_1 + m_p + m_{p^2} + \ldots \geq i$. Choose a map of commutative rings $\phi : R \to \pi_* \mathrm{MU}$ with the following properties:

(1) If $i + 1$ is not a power of p, then $\phi(u_i)$ is an element of $\pi_{2i} \mathrm{MU}$ representing $c_i \in \mathrm{gr}_0(\pi_{2i} \mathrm{MU})$.

(2) If $i + 1 = p^v$, then $\phi(u_i)$ is an element of $F^1 \pi_{2i} \mathrm{MU}$ representing $c_i \in \mathrm{gr}_1 \pi_{2i} \mathrm{MU}$.

Then ϕ is compatible with both the grading and the filtrations on R and $\pi_* \mathrm{MU}$. It follows from Proposition 2 that ϕ induces an isomorphism of associated graded rings $\mathrm{gr}\, R \to \mathrm{gr}(\pi_* \mathrm{MU})$. It follows by induction on i that ϕ induces an isomorphism of quotients $R/F^i R \to (\pi_* \mathrm{MU})/F^i \pi_* \mathrm{MU}$. Passing to the inverse limit, we get an isomorphism of graded rings

$$\mathbf{Z}_p[u_1, u_2, \ldots] \simeq \varprojlim R/F^i R \simeq \varprojlim (\pi_* \mathrm{MU})/F^i \pi_* \mathrm{MU} \simeq (\pi_* \mathrm{MU})^\vee.$$

Here $\vee$ denotes the functor of p-adic completion (applied in each graded degree).

We are now ready to prove Quillen's theorem:

Theorem 4 (Quillen)**.** *Let $\theta : L \to \pi_* \mathrm{MU}$ be the ring homomorphism classifying the formal group law coming from the universal complex orientation on* MU*. Then θ is an isomorphism.*

Proof. We have already seen that the composite map $L \xrightarrow{\theta} \pi_* \mathrm{MU} \to \mathrm{H}_*(\mathrm{MU}; \mathbf{Z}) \simeq \mathbf{Z}[b_1, \ldots]$ is an injection. We show that θ is surjective. Since each homotopy group of MU is finitely generated, it will suffice to show that θ induces a surjection $L^\vee \to (\pi_* \mathrm{MU})^\vee$ after p-adically completing at every prime p.

Using Lazard's theorem we can identify $L^\vee \simeq \mathbf{Z}_p[t_1, t_2, \ldots]$, and the above analysis gives an isomorphism of graded rings $(\pi_* \mathrm{MU})^\vee \simeq \mathbf{Z}_p[u_1, u_2, \ldots]$. Let I denote the ideal of $L^\vee$ generated by homogeneous elements of positive degree and let $K \subseteq (\pi_* \mathrm{MU})^\vee$, $J \subseteq \mathbf{Z}_2[b_1, b_2, \ldots]$ be defined similarly. As in the proof of Lazard's theorem, it will suffice to show that the map $I/I^2 \to K/K^2$ is surjective in each degree. In each degree, we have identifications $(I/I^2)_{2n} \simeq \mathbf{Z}_p t_n$ and $(K/K^2)_{2n} \simeq \mathbf{Z}_p u_n$, so that $\theta(t_n) = \lambda u_n + \text{ decomposables }$. We wish to prove that λ is a p-adic unit. The Hurewicz map carries u_n to $\lambda' b_n + \text{ decomposables }$. In the proof of Lazard's theorem, we saw that

$$\lambda\lambda' = \begin{cases} p & \text{if } n+1 = p^v \\ 1 & \text{otherwise.} \end{cases}$$

If $n+1$ is not a power of p, it follows immediately that λ is a p-adic unit. If $n+1 = p^v$, then we need to work a little harder: namely, we need to show that λ' is divisible by p. It will suffice to show that the image of u_n vanishes in the ring $\mathrm{H}_*(\mathrm{MU}; \mathbf{F}_p) \simeq \mathbf{F}_p[b_1, b_2, \ldots]$. This is equivalent to the requirement that u_n have Adams filtration ≥ 1, which is true by construction. □

Let us now return to the construction of the Adams spectral sequence. Let X be an arbitrary spectrum. Since the complex bordism spectrum MU is coherently associative, we can define a cosimplicial spectrum $X^\bullet$ by the formula $X^n = X \otimes \mathrm{MU}^{\otimes n+1}$. If X is connective, then one can show that the map $X \to \mathrm{Tot}\, X^\bullet \simeq \varprojlim \mathrm{Tot}^n X^\bullet$ is an equivalence. We therefore obtain a spectral sequence $\{E_r^{a,b}, d_r\}$ which carries information about the homotopy groups of X. This spectral sequence is called the *Adams-Novikov spectral sequence.* It has slightly different behavior than the classical Adams spectral sequence: since $\pi_0 \mathrm{MU} \simeq \pi_0 S \simeq \mathbf{Z}$, the convergence is very fast. Namely, if we define $F^n \pi_* X$ to be the kernel of the map

$$\pi_* X \to \pi_* \mathrm{Tot}^{n+1} X^\bullet,$$

then we have for every integer $n \geq 0$ a *finite* filtration

$$0 = F^{n+1}\pi_n X \subseteq F^n \pi_n X \cdots \subseteq F^1 \pi_n X \subseteq F^0 \pi_n X = \pi_n X.$$

The E_1-term of the Adams-Novikov spectral sequence is given by the chain complex of graded abelian groups

$$\mathrm{MU}_*(X) \to (\mathrm{MU} \otimes \mathrm{MU})_*(X) \to (\mathrm{MU} \otimes \mathrm{MU} \otimes \mathrm{MU})_* X \to \cdots$$

We would like to understand this chain complex in algebro-geometric terms.

For any complex-oriented cohomology theory E, we have a canonical isomorphism $\pi_*(E \otimes \mathrm{MU}) \simeq (\pi_* E)[b_1, b_2, \ldots]$. Assuming that the homotopy groups of E are concentrated in even degrees (so that $\pi_* E$ is a commutative ring R), we conclude that $\mathrm{Spec}\, \pi_*(E \otimes \mathrm{MU})$ is an infinite dimensional affine space over $\mathrm{Spec}\, R$: more precisely, it is the affine space parametrizing all coordinates

$$g(t) = t + b_1 t^2 + b_2 t^3 + \cdots$$

on the formal power series ring $R[[t]]$ which agree with the standard coordinate to first order.

Let $G = \mathrm{Spec}\, \mathbf{Z}[b_1, b_2, \ldots]$ be the scheme whose R-points are power series $g(t) = t + b_1 t^2 + b_2 t^3 + \cdots \in R[[t]]$, regarded as a group under composition. We conclude that $\pi_*(E \otimes \mathrm{MU})$ is the ring of functions on the affine scheme $G \times \mathrm{Spec}\, \pi_* E$. In particular, Quillen's theorem gives $\mathrm{Spec}\, \pi_*(\mathrm{MU} \otimes \mathrm{MU}) \simeq G \times \mathrm{Spec}\, L$. Here the two natural inclusions $\mathrm{MU} \to \mathrm{MU} \otimes \mathrm{MU} \leftarrow \mathrm{MU}$ induce a pair of maps $G \times \mathrm{Spec}\, L \to \mathrm{Spec}\, L$. In concrete terms, this means that given a formal group $f(x, y) \in R[[x, y]]$ and a power series $g(t) = t + b_1 t^2 + \ldots$, we can naturally construct *two* formal groups over R: the first is given by f itself, and the second by the formula $gf(g^{-1}(x), g^{-1}(y))$. In other words, the group G of coordinate changes acts on the moduli space $\operatorname{Spec} L$ parametrizing formal groups.

More generally, the same reasoning shows that $\pi_* \mathrm{MU}^{\otimes n+1}$ can be identified with the ring of functions on the product scheme $G^n \times \operatorname{Spec} L$. In particular, the cosimplicial spectrum $\mathrm{MU}^{\otimes \bullet+1}$ gives rise to a simplicial scheme $\operatorname{Spec} \pi_*(\mathrm{MU}^{\otimes \bullet+1})$, which encodes the canonical action of G on $\operatorname{Spec} L$.

Definition 5. We let $\mathcal{M}_{\mathrm{FG}}^s$ denote the quotient stack $\operatorname{Spec} L/G$. We will refer to $\mathcal{M}_{\mathrm{FG}}^s$ as the moduli stack of formal groups and *strict* isomorphisms.

More precisely, $\mathcal{M}_{\mathrm{FG}}^s$ is a functor which assigns to each commutative ring R the category whose objects are formal groups $f \in R[[x, y]]$, where a morphism from f to f' is a power series $g(t) = t + b_1 t^2 + \ldots$ such that $f(g(x), g(y)) = gf'(x, y)$. Here the word "strict" refers to the requirement that $g(t)$ have leading coefficient t. (One can show that $\mathcal{M}_{\mathrm{FG}}^s$ is in fact a stack: that is, the groupoids defined above satisfy descent with respect to the flat topology.)

Now suppose that X is an arbitrary spectrum. It is clear that $\pi_* X^n = \pi_*(X \otimes \mathrm{MU}^{\otimes n+1})$ is a module over the commutative ring $\pi_* \mathrm{MU}^{\otimes n+1}$, and can therefore be identified with a quasi-coherent sheaf on the affine scheme $\operatorname{Spec} L \times G^n$. These quasi-coherent sheaves are compatible with one another under base change. In the language of algebraic stacks, this means:

- Let X be any spectrum. Then $\pi_* X$ can be regarded as a quasi-coherent sheaf $\mathcal{F}_X$ on the quotient stack $\mathcal{M}_{\mathrm{FG}}^s$.

Put more concretely, the abelian group $\mathrm{MU}_*(X)$ is a module over $\pi_* \mathrm{MU} \simeq L$, and can therefore be regarded as a quasi-coherent sheaf on $\operatorname{Spec} L$. This quasi-coherent sheaf carries an action of the group scheme G defined above, compatible with the action of G on $\operatorname{Spec} L$.

The cochain complex

$$\mathrm{MU}_*(X) \to (\mathrm{MU} \otimes \mathrm{MU})_*(X) \to (\mathrm{MU} \otimes \mathrm{MU} \otimes \mathrm{MU})_* X \to \cdots$$

now admits a natural interpretation: it is simply the standard complex for computing the cohomology of $\mathcal{M}_{\mathrm{FG}}^s$ with coefficients in $\mathcal{F}_X$. In other words, we have the following:

Proposition 6. *Let X be any spectrum. The second page of the Adams-Novikov spectral sequence is given by*

$$E_2^{*b} \simeq \mathrm{H}^b(\mathcal{M}_{\mathrm{FG}}^s; \mathcal{F}_X) \simeq \mathrm{H}^b(G; \mathrm{MU}_*(X)).$$