Math 221: Problem Set 4

(1) Let $R$ be a commutative ring. An element $e \in R$ is said to be idempotent if $e^2 = e$. Show that if $I \subseteq R$ is a radical ideal, then the closed subset $V(I) = \{ p \in \text{Spec } R : I \subseteq p \}$ of $\text{Spec } R$ is open if and only if $I$ is the radical of $(e)$ for some idempotent element $e \in R$. Conclude that $\text{Spec } R$ is connected if and only if the only idempotent elements of $R$ are 0 and 1.

(2) Let $R$ be a commutative ring containing elements $a_1, \ldots, a_n \in R$ which generate the unit ideal (1). Let $f : M \rightarrow N$ be an $R$-module homomorphism. Show that if each of the induced maps $M[\frac{1}{a_i}] \rightarrow N[\frac{1}{a_i}]$ is injective, surjective, or an isomorphism, then $f$ has the same property.

(3) Let $K$ be a field and let $v : K \rightarrow \mathbb{R} \cup \{\infty\}$ be a valuation on $K$, where $\mathbb{R}$ denotes the field of real numbers. For $x, y \in K$, define $d(x, y) = \exp -v(x - y)$ (here we adopt the convention that $\exp -\infty = 0$, so that $d(x, x) = 0$). Prove that $(K, d)$ is a metric space. Let $K^\vee$ be the completion of $K$ with respect to the metric $d$. Prove that the addition and multiplication on $K$ have a unique continuous extension to addition and multiplication maps $K^\vee \times K^\vee \rightarrow K^\vee$, which endow $K^\vee$ with the structure of a field.

(4) Let $v : K \rightarrow \mathbb{R} \cup \{\infty\}$ be as in (3), and let $R = \{ x \in K : v(x) \geq 0 \}$. Show that every nonzero prime ideal of $R$ is maximal.

(5) Let $n$ be a square-free odd integer. Prove that the ring $\mathbb{Z}[\sqrt{n}]$ is integrally closed if $n$ is congruent to 3 mod 4. Describe its integral closure when $n$ is congruent to 1 mod 4.