Math 221: Problem Set 11

(1) Let $f : A \to B$ be a map of commutative rings, let $\mu$ denote the multiplication map $\mu : B \otimes_A B \to B$, and let $I$ be the kernel of $\mu$. Show that the construction

$$b \mapsto 1 \otimes b - b \otimes 1$$

determines an $A$-linear derivation from $B$ into $I/I^2$, which induces an isomorphism of $B$-modules $\Omega_{B/A} \cong I/I^2$.

(2) Let $k$ be a field and let $\overline{k}$ be its algebraic closure. Show that $\Omega_{\overline{k}/k} \cong 0$ (note that $\overline{k}$ need not be a separable extension of $k$).

(3) Let $R$ be a complete local Noetherian ring having maximal ideal $m$. Assume that the field $R/m$ has characteristic zero. Show that there exists a subfield $k \subseteq R$ such that the composite map $\alpha : k \to R \to R/m$ is an isomorphism. (Hint: use Zorn’s lemma to choose $k$ maximal among subfields of $R$, and use Hensel’s lemma to show that $\alpha$ is an isomorphism).

(4) Let $f : A \to B$ and $g : A \to A'$ be maps of commutative rings, and set $B' = B \otimes_A A'$. Construct an isomorphism $\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_B B'$. 