Lecture 5: Norms

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Our goal in this lecture is to describe another way of thinking about some of the rings appearing in the previous lecture. First, we review some topological algebra.

**Definition 1.** Let $V$ be a topological vector space over $\mathbb{Q}_p$. We say that $V$ is a $p$-adic Banach space if there exists an open $\mathbb{Z}_p$-submodule $V_0 \subseteq V$, which is closed under addition, such that $V_0$ is $p$-adically complete as an abelian group and satisfies $V = V_0[\frac{1}{p}]$.

**Example 2.** Fix a non-archimedean norm $|\cdot|_{\mathbb{Q}_p}$ on $\mathbb{Q}_p$, compatible with the usual topology. For example, we can take the usual $p$-adic norm, characterized by $|p|_{\mathbb{Q}_p} = \frac{1}{p}$; however, it will be convenient not to assume this.

Let $V$ be a vector space over $\mathbb{Q}_p$. We define a norm on $V$ to be a function $|\cdot|_V : V \to \mathbb{R}_{\geq 0}$ satisfying

$$|\lambda v|_V = |\lambda|_{\mathbb{Q}_p} \cdot |v|_V \quad |v + w|_V \leq \max(|v|_V, |w|_V)$$

(this is sometimes called a pre-norm, with the term norm reserved for the case where $|v|_V = 0 \Rightarrow v = 0$).

Any norm on $V$ equips $V$ with the structure of a (pre)metric space, with metric $d(v,w) = |v - w|_V$. If $V$ is separated and complete with respect to this metric, then it is a $p$-adic Banach space (take $V_0 = \{v \in V : |v|_V \leq 1\}$ to be the “unit ball” of $V$).

**Remark 3.** Every $p$-adic Banach space $V$ can be obtained from the construction of Example 2. Let $V_0 \subseteq V$ be an open $\mathbb{Z}_p$-module which is $p$-adically complete. We can then define a map $|\cdot|_V : V \to \mathbb{R}_{\geq 0}$ by the formula

$$|v|_V = \inf\{|\lambda|_{\mathbb{Q}_p} : v \in \lambda \cdot V_0\};$$

this is a norm on $V$, having $V_0$ as the unit ball.

**Example 4.** Let $V$ be a vector space over $\mathbb{Q}_p$ equipped with a pair of norms $|\cdot|_V$ and $|\cdot|'_V$ (possibly with respect to different choices of absolute value $|\cdot|_{\mathbb{Q}_p}$ and $|\cdot|_{\mathbb{Q}_p}'$). We can then regard $V$ as a metric space with respect to the metric $d(v,w) = |v - w|_V + |v - w|'_V$. If $V$ is complete with respect to this metric, then it is a $p$-adic Banach space (the intersection of unit balls $V_0 = \{v \in V : |v|_V \leq 1\}$ and $|v|'_V \leq 1\}$ satisfies the requirements of Definition 1).

Alternatively, in the case $|\cdot|_{\mathbb{Q}_p} = |\cdot|_{\mathbb{Q}_p}'$ (which we can always arrange by raising to an appropriate power), we can equip $V$ with the norm $v \mapsto |v|_V + |v|'_V$.

**Example 5.** Let $K$ be any completely valued field of characteristic zero and residue characteristic $p$. Then $K$ is a $p$-adic Banach space.

**Example 6.** Let $V_0$ be any abelian group which is $p$-adically complete and $p$-torsion free. Then $V_0$ has the structure of a module over the ring $\mathbb{Z}_p$, and the tensor product $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} V_0 = V_0[\frac{1}{p}]$ can be regarded as a $p$-adic Banach space (by equipping it with the topology where the subsets $p^nV_0$ form a neighborhood basis of the identity).
Example 7. Let $M$ be an abelian group which is $p$-torsion free. We can then apply the construction of Example 6 to the $p$-adic completion $\hat{M} = \lim\limits_{\rightarrow} M/p^n M$ to obtain a $p$-adic Banach space $\hat{M}$. 

Example 8. Let $V$ be a $\mathbb{Q}_p$-vector space equipped with a norm. Then the completion of $V$ (as a metric space) is a $\mathbb{Q}_p$-Banach space.

Examples 7 and 8 are related. If $V$ is a $\mathbb{Q}_p$-vector space equipped with a norm, then the unit ball $V_0 = \{v \in V : |v|_V \leq 1\}$ is a $p$-torsion free abelian group. The completion of $V$ with respect to its norm can then be identified with $\hat{V}_0[\frac{1}{p}]$, where $\hat{V}_0$ is the $p$-adic completion of $V_0$.

Variant 9. Suppose that $V$ is equipped with a pair of norms $|\bullet|_V$ and $|\bullet|_{V'}$. Then the completion of $V$ with respect to the metric $d(v,w) = |v-w|_V + |v-w|_{V'}$ is given by $\hat{V}_0[\frac{1}{p}]$, where $V_0 = \{v \in V : |v|_V \leq 1, |v|_{V'} \leq 1\}$.

Let us now turn to the example of interest to us. Fix a perfectoid field $C^\flat$, with valuation ring $\mathcal{O}_C^\flat$ and set $\mathcal{A}_{\text{inf}} = W(\mathcal{O}_C^\flat)$. Fix an element $\pi \in C^\flat$ satisfying $0 < |\pi|_{C^\flat} < 1$ and consider the localization $\mathcal{A}_{\text{inf}}[\frac{1}{\pi}, \frac{1}{\pi}]$. Every element of this ring admits a Teichmüller expansion

$$\sum_{n \gg -\infty} c_n p^n$$

where the coefficients $c_n \in C^\flat$ are bounded.

Definition 10. [Gauss Norms] Fix a real number $0 < \rho < 1$. For each element $f = \sum_{n \gg -\infty} c_n p^n \in \mathcal{A}_{\text{inf}}[\frac{1}{\pi}, \frac{1}{\pi}]$, we define

$$|f|_\rho = \sup \{|c_n|_{C^\flat} \cdot \rho^n\}.$$ 

Remark 11. In the situation of Definition 10, the real numbers $|c_n|_{C^\flat} \cdot \rho^n$ vanish for $n \ll 0$ and decay exponentially as $\rho \to \infty$. Consequently, the supremum $\sup \{|c_n|_{C^\flat} \cdot \rho^n\}$ is finite and realized by finitely many values of $n$.

Notation 12. We let $Y$ denote the set of all isomorphism classes of characteristic zero untilts $(K, \iota)$ of $C^\flat$. We will use the letter $y$ to denote a typical point of $Y$, given by an untilt $(K, \iota)$ of $C^\flat$. For every such point $y$, we have a surjective ring homomorphism

$$\theta_y : \mathcal{A}_{\text{inf}} \to \mathcal{O}_K, \quad \sum_{n \geq 0} c_n p^n \mapsto \sum_{n \geq 0} c_n p^n$$

which extends to a ring homomorphism $\mathcal{A}_{\text{inf}}[\frac{1}{\pi}, \frac{1}{\pi}] \to K$. We denote the value of this homomorphism on an element $f \in \mathcal{A}_{\text{inf}}[\frac{1}{\pi}, \frac{1}{\pi}]$ by $f(y) \in K$.

Given $0 < a \leq b < 1$, we let $Y_{[a,b]} \subseteq Y$ denote the subset consisting of those points $y = (K, \iota)$ satisfying $a \leq |p|_K \leq b$.

Remark 13. Let $y = (K, \iota)$ be a point of $Y$ and let $\rho = |p|_K$. Then, for every element $f = \sum_{n \gg -\infty} c_n p^n \in \mathcal{A}_{\text{inf}}[\frac{1}{\pi}, \frac{1}{\pi}]$, we have

$$|f(y)|_K = |\sum_{n \gg -\infty} c_n p^n|_K \leq \sup \{|c_n|_K \cdot |p|_K\} = \sup \{|c_n|_{C^\flat} \cdot \rho^n\} = |f|_\rho.$$ 

Remark 14. Let $f = \sum_{n \gg -\infty} c_n p^n$ be an element of $\mathcal{A}_{\text{inf}}[\frac{1}{\pi}, \frac{1}{\pi}]$. We will say that a real number $\rho \in (0,1)$ is generic for $f$ if the supremum $\sup \{|c_n|_{C^\flat} \cdot \rho^n\}$ is achieved exactly once. That is, $\rho$ is generic for $f$ if there is an integer $n$ such that $|f|_\rho = |c_n|_{C^\flat} \cdot \rho^n$, and for all integers $m \neq n$ we have $|c_m|_{C^\flat} \cdot \rho^m < |f|_\rho$. In this case, if $y = (K, \iota)$ is a point of $Y$ satisfying $|p|_K = \rho$, the inequality of Remark 13 can be replaced by an equality $|f|_\rho = |f(y)|_K$. 

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Exercise 15. Let \( f \) be an element of \( A_{\text{inf}}[\frac{1}{p}, \frac{1}{|p|}] \). Show that the set
\[
\{ \rho \in (0, 1) : \rho \text{ is not generic for } f \}
\]
is a discrete subset of \((0, 1)\). In other words, if \( \rho \) is not generic for \( f \), then \( \rho \pm \epsilon \) will be generic for \( f \) for all sufficiently small \( \epsilon \neq 0 \).

Proposition 16. For each \( 0 < \rho < 1 \), the map \( \| \cdot \|_\rho : A_{\text{inf}}[\frac{1}{p}, \frac{1}{|p|}] \to \mathbb{R}_{\geq 0} \) is a norm (in the sense of Example 2), compatible with the norm on \( \mathbb{Q}_p \) satisfying \( |p|_{\mathbb{Q}_p} = \rho \).

Proof. We first show that \( |f + g|_\rho \leq \max(|f|_\rho, |g|_\rho) \). Write \( f + g = \sum_{n \geq -\infty} [c_n]p^n \). Suppose that the following conditions are satisfied:

(*) The real number \( \rho \) is generic for \( f \) and belongs to the value group of \( C^\circ \).

In this case, we can choose a point \( y = (K, \iota) \in Y \) satisfying \( |p|_K = \rho \) (for example, by taking \( O_K = A_{\text{inf}}/(\{c\} - p) \), where \( c \in C^\circ \) is any element satisfying \( |c|_{C^\circ} = \rho \)). Remark 14 then gives
\[
|f + g|_\rho = |(f + g)(y)|_K \leq \max(|f(y)|_K, |g(y)|_K) \leq \max(|f|_\rho, |g|_\rho).
\]

It follows from Exercise 15 that the collection of real numbers \( \rho \) satisfying (*) is dense in \((0, 1)\). Consequently, it follows by continuity that \( |f + g|_\rho \leq \max(|f|_\rho, |g|_\rho) \) for all \( \rho \in (0, 1) \).

It follows for that each \( f \in A_{\text{inf}}[\frac{1}{p}, \frac{1}{|p|}] \) and each integer \( n \), we have \( |nf|_\rho \leq |f|_\rho \). By a continuity argument, we conclude that \( |\lambda f|_\rho \leq |f|_\rho \) for each \( \lambda \in \mathbb{Z}_p \). If \( \lambda \) is an invertible element of \( \mathbb{Z}_p \), then the same argument gives \( |f|_\rho \leq |\lambda f|_\rho \), so that \( |\lambda f|_\rho = |f|_\rho = |\lambda|_{\mathbb{Q}_p} \cdot |f|_\rho \). Since every nonzero element of \( \mathbb{Q}_p \) factors as \( p^n u \), where \( u \) is an invertible element of \( \mathbb{Z}_p \), we are reduced to checking the identity \( |\lambda f|_\rho = |\lambda|_{\mathbb{Q}_p} \cdot |f|_\rho \) in the case \( \lambda = p \): that is, the identity \( |pf|_\rho = |p| \cdot |f|_\rho \). This follows immediately from the definition.

Variant 17. For every pair of elements \( f, g \in A_{\text{inf}}[\frac{1}{p}, \frac{1}{|p|}] \), we have \( |f \cdot g|_\rho = |f|_\rho \cdot |g|_\rho \).

Proof. Assume first that the following condition is satisfied:

(*) The element \( \rho \) is generic for \( f, g \), and \( f \cdot g \), and belongs to the value group of \( C^\circ \).

As in the proof of Proposition 16, we can choose a point \( y = (K, \iota) \in Y \) satisfying \( |p|_K = \rho \). In this case, Remark 13 gives
\[
|f \cdot g|_\rho = |(f \cdot g)(y)|_K = |f(y)|_K \cdot |g(y)|_K = |f|_\rho \cdot |g|_\rho.
\]

We conclude by observing that the collection of real numbers \( \rho \in (0, 1) \) satisfying (*) is dense, so by continuity we have an equality \( |f g|_\rho = |f|_\rho \cdot |g|_\rho \) for all \( \rho \in (0, 1) \).

Proposition 18. Suppose that \( a \) and \( b \) belong to the value group of \( C^\circ \), so that we can choose elements \( \pi_a, \pi_b \in C^\circ \) satisfying \( |\pi_a|_{C^\circ} = a \) and \( |\pi_b|_{C^\circ} = b \). Then the intersection of unit balls
\[
V_0 = \{ f \in A_{\text{inf}}[\frac{1}{p}, \frac{1}{|p|}] : |f|_a \leq 1, |f|_b \leq 1 \}
\]
is the subring \( A_{\text{inf}}[\frac{\pi_a}{p}, \frac{\pi_b}{|p|}] \) of the previous lecture.

Proof. It follows from Proposition 16 and Variant 17 that \( V_0 \) is a subring of \( A_{\text{inf}}[\frac{1}{p}, \frac{1}{|p|}] \). This subring clearly contains \( A_{\text{inf}} \): note that if \( f = \sum_{n \geq 0} [c_n]p^n \) belongs to \( A_{\text{inf}} \), then we automatically have
\[
|f|_\rho = \sup_{n \geq 0} |[c_n]|_{C^\circ} \cdot \rho^n \leq 1
\]
for any $0 < \rho < 1$. Moreover, it also contains $\left[ \frac{\pi_a}{p} \right]$ and $\left[ \frac{p}{\pi_b} \right]$, by virtue of the equalities

$$\left| \frac{\pi_a}{p} \right|_a = 1 \quad \left| \frac{\pi_a}{p} \right|_b = \frac{a}{b} < 1$$

$$\left| \frac{p}{\pi_b} \right|_a = \frac{a}{b} < 1 \quad \left| \frac{p}{\pi_b} \right|_b = 1.$$  

This shows that $A_{\inf}[\frac{\pi_a}{p}, \frac{p}{\pi_b}]$ is contained in $V_0$.

We now prove the reverse containment. Suppose that $f = \sum_{n \gg -\infty} [c_n]p^n$ satisfies $|f|_a \leq 1$ and $|f|_b \leq 1$; we wish to show that $f$ belongs to $A_{\inf}[\frac{\pi_a}{p}, \frac{p}{\pi_b}]$. By assumption, the absolute values $|c_n|_{C^\circ}$ are bounded above. We may therefore choose some integer $m \gg 0$ such that each product $\pi_n a c_n$ belongs to $C^\circ$. We then have

$$f = (\sum_{n \leq m} [c_n]p^n) + (\sum_{n \geq 0} [c_n+m\pi_b^m]p^n) \left( \frac{p}{\pi_b} \right)^m$$

where the second summand belongs to $A_{\inf}[\frac{p}{\pi_b}]$ (and therefore also to the unit ball of $A_{\inf}[\frac{\pi_a}{p}, \frac{p}{\pi_b}]$). Subtracting, we can reduce to the case where the Teichmüller expansion of $f$ is finite.

Our assumption that $|f|_a \leq 1$ and $|f|_b \leq 1$ guarantees that, for each integer $n$, we have

$$|c_n|_{C^\circ} \cdot a^n \leq 1 \quad |c_n|_{C^\circ} \cdot b^n \leq 1,$$

so that $c_n a^n$ and $c_n b^n$ belong to $0^C$. For $n \leq 0$, this implies that $[c_n]p^n = [c_n a^n]((\frac{\pi_a}{p})^{-n}$ belongs to $A_{\inf}[\frac{\pi_a}{p}]$. For $n \geq 0$, we instead learn that $[c_n]p^n = [c_n b^n]((\frac{p}{\pi_b})^n$ belongs to $A_{\inf}[\frac{p}{\pi_b}]$. It follows that $f$ belongs to the ring $A_{\inf}[\frac{\pi_a}{p}, \frac{p}{\pi_b}]$, as desired.

\[ \square \]

**Corollary 19.** Suppose that $a$ and $b$ belong to the value group of $C^\circ$. Then the ring $B_{[a,b]}$ of the previous lecture can be identified with the completion of $A_{\inf}[\frac{1}{p}, -\frac{1}{|\pi|}]$ with respect to the pair of norms $|\bullet|_a$ and $|\bullet|_b$.

We will henceforth use this Corollary to extend the definition of $B_{[a,b]}$ to the case where $a$ and $b$ do not necessarily belong to the value group of $C^\circ$. Note that if $y = (K, i) \in Y$ is an untilt satisfying $a \leq |p|_K \leq b$, then Remark 13 implies that the homomorphism

$$A_{\inf}[\frac{1}{p}, -\frac{1}{|\pi|}] \to K \quad f \mapsto f(y)$$

admits a continuous extension $B_{[a,b]} \to K$, which we will also denote by $f \mapsto f(y)$. 

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